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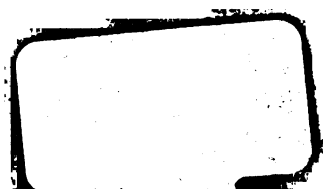
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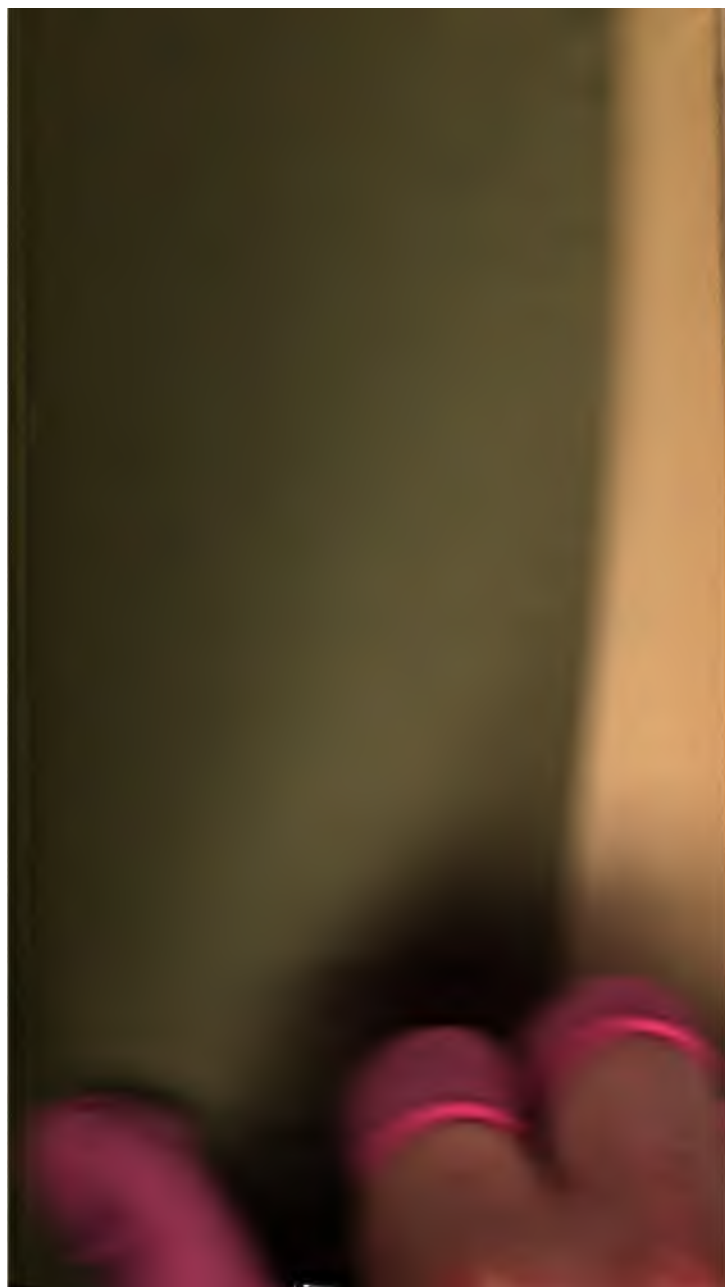
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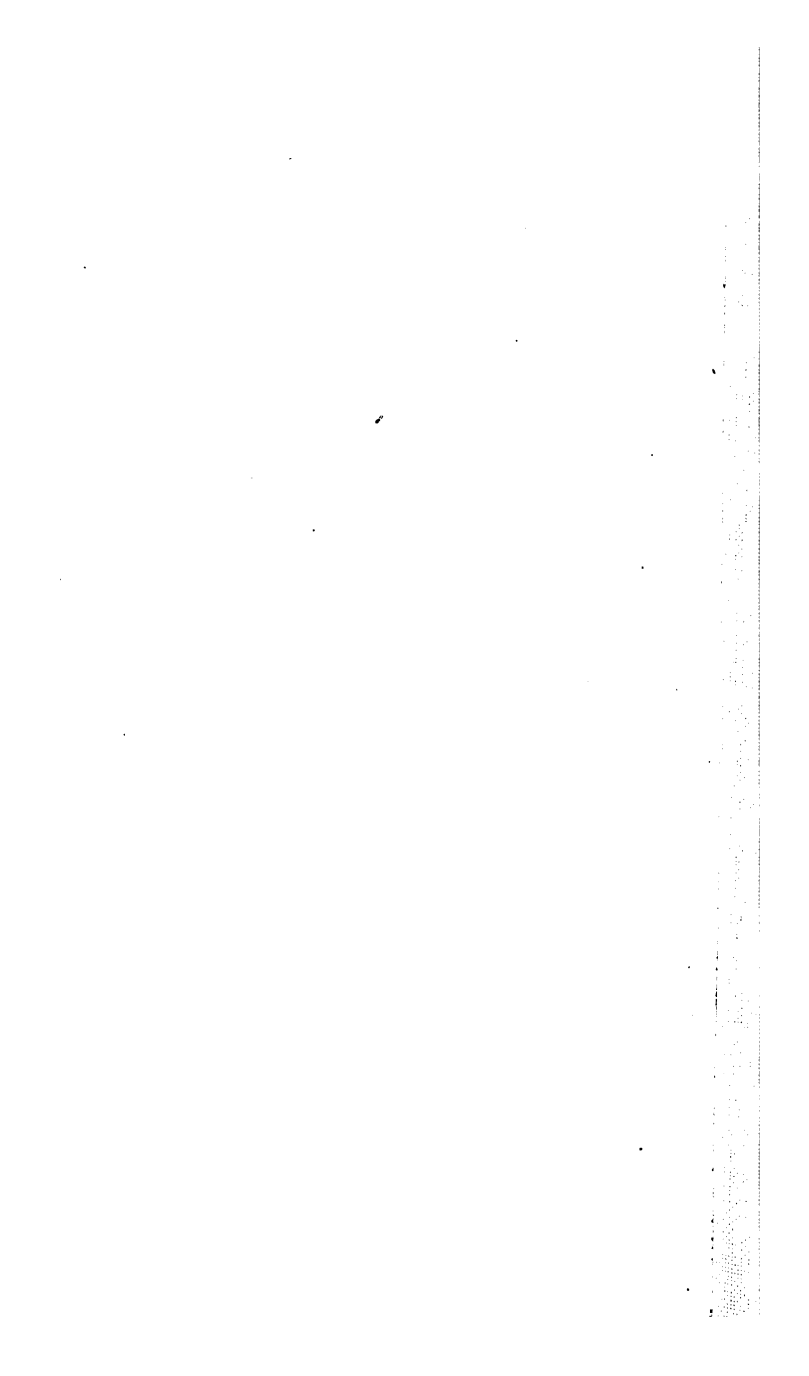
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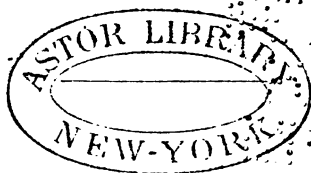




ELEMENTS  
OF  
ALGEBRA.

BY  
JAMES HADDON, M.A.

ARITHMETICAL MASTER  
AND SECOND MATHEMATICAL MASTER OF KING'S COLLEGE SCHOOL,  
LONDON;  
AUTHOR OF "RUDIMENTARY ARITHMETIC."



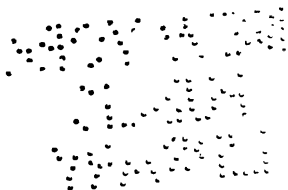
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1850.



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## PREFACE.

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EVERY one who has made some advancement in the study of this important branch of mathematical science, must have found that, until he had acquired a knowledge of the nature of Equations, his progress was uninteresting, and, consequently, unsatisfactory to himself, because he met with many things which were not easily comprehensible to him, and because the application of the principles he was endeavouring to comprehend was so remote that he would frequently be led to question the utility of this science. This circumstance has frequently operated as a serious discouragement to persons entering upon the study of Algebra.

The arrangement of the following pages is such that, as soon as the student shall have made himself so far acquainted with the first principles as to be able to perform the ordinary operations of Addition, Subtraction, Multiplication, and Division, he shall be invited to consider the nature of an Equation, and to apply his knowledge, however small, to the *solution of a Problem*. Again, when he has become conversant with Algebraic Fractions, he will be presented with other equations and problems, which it is hoped his increased knowledge will enable him to encounter with confidence and success.

One of the greatest of modern mathematicians has said that "In ediscendis scientiis exempla plus prosunt quam præcepta;" and any one who has been long practically engaged in the work of education will readily agree with him. With a view to assist the student, explanations and solutions of many examples and problems are given in all parts of this work.

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# ALGEBRA.

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## CHAPTER I.

I. ALGEBRA is a science in which numerical magnitude in general is represented by symbols and signs.

The *symbols* are  $a, b, c$ , &c., or  $\alpha, \beta, \gamma$ , &c., to represent *known* quantities, and  $x, y, z$ , &c., or  $\theta, \phi, \psi$ , &c., to represent *unknown* quantities.

The signs are

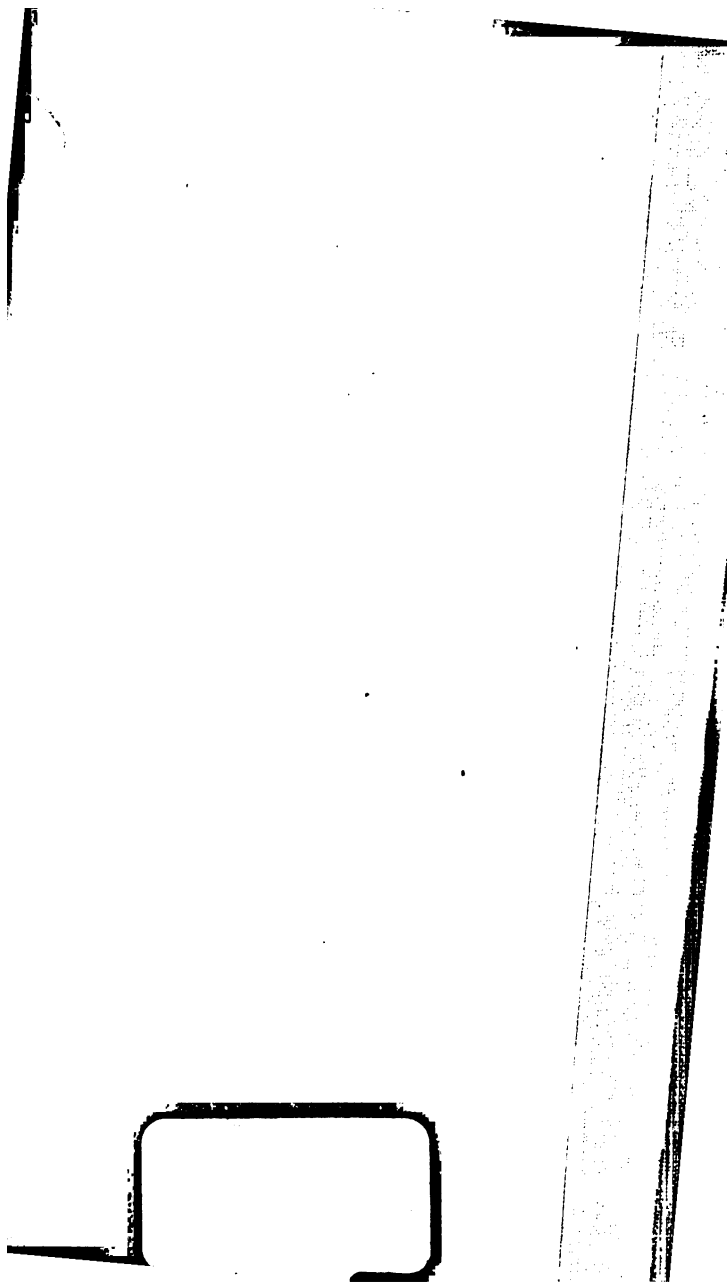
= *equal*.

+ *plus*, for *Addition*; thus,  $a + b$  = the *sum* of  $a$  and  $b$ , just as  $3 + 5$  = the sum of 3 and 5 = 8.

— *minus*, for *Subtraction*; thus  $b - c$  = the excess of  $b$  above  $c$ , just as  $5 - 2$  = the excess of 5 above 2, = 3.

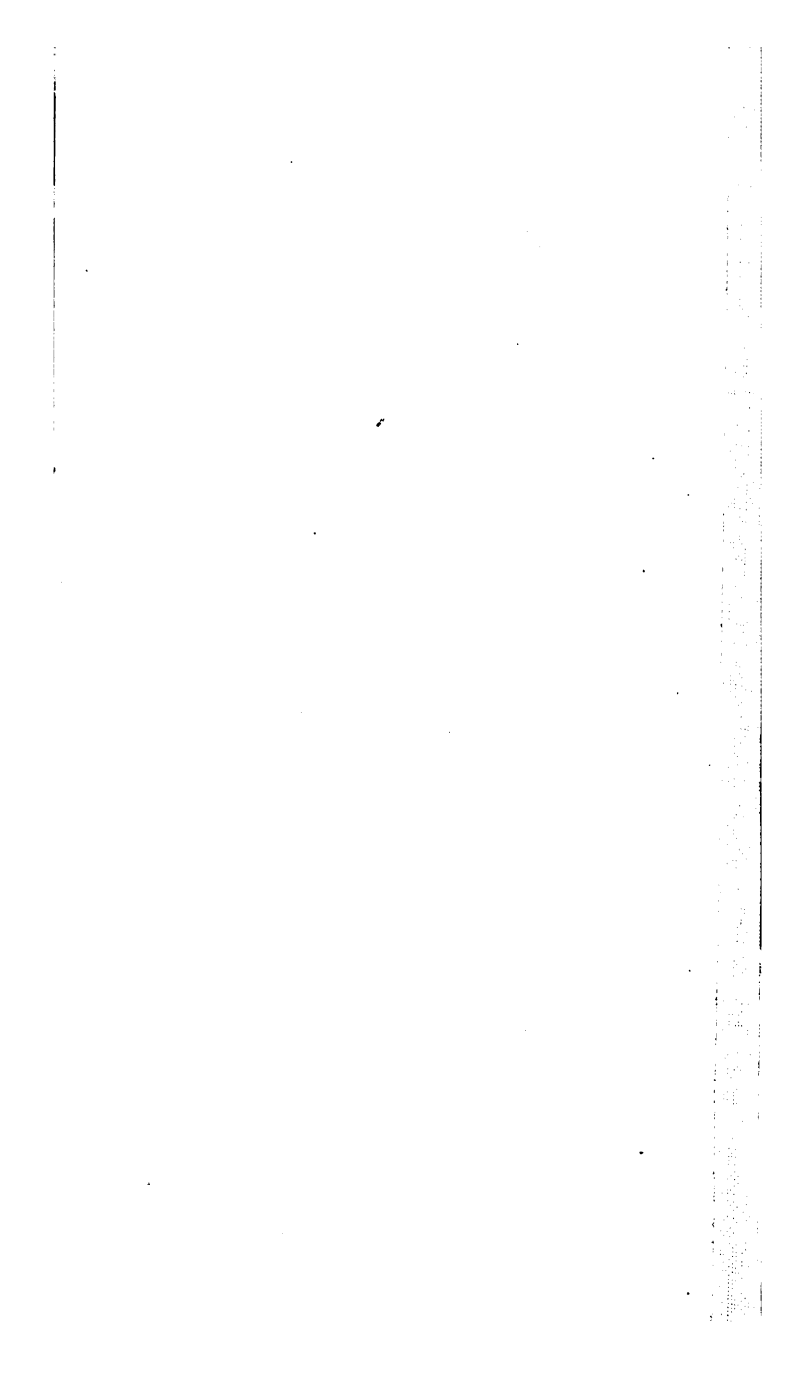
When it is not known which of two quantities, as  $x$  and  $y$ , is the greater, their *difference* is expressed thus,  $x \sim y$ .

$\times$  or  $\cdot$  for *Multiplication*, thus  $2c \times 3d$  = the *product* of twice  $c$  and thrice  $d$ . The 2 is called the *coefficient* of  $c$ , and the 3 the coefficient of  $d$ . In the expressions  $ax$  and  $bx$ , that is,  $a \times x$  and  $b \times x$ ,  $a$  and  $b$  are the *literal* coefficients of  $x$ . The product is sometimes expressed thus,  $(a + b) \cdot (c - d)$  or  $(a + b) (c - d)$  or  $\overline{a + b} \cdot \overline{c - d}$ , which means the sum of  $a$  and  $b$  *multiplied* by the difference of  $c$  and  $d$ . The expression  $ab + c$  means that  $a$  is *multiplied* by  $b$ , and then  $c$  is added to the



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## SUBTRACTION.

### *Examples.*

1. From  $4a - 8b - 2c$  subtract  $a + 2b - 5c$ .

$$4a - 8b - 2c$$

$$a + 2b - 5c$$


---

$$3a - 10b + 3c \text{ Ans.}$$

$$4a - 8b - 2c \text{ Proof.}$$


---

Here we say  $+4a - 1a = +3a$   
 $-8b - 2b = -10b$   
 $-2c + 5c = +3c$ .

2. From  $2mx^2 - nx - 3py + r$  subtract  $5mx^2 - 4nx + py + q$ .

$$2mx^2 - nx - 3py + r$$

$$5mx^2 - 4nx + py + q$$


---

$$-3mx^2 + 8nx - 4py + r - q \text{ Ans.}$$


---

Here we say  
 $+2mx^2 - 5mx^2 = -3mx^2$   
 $-nx + 4nx = +3nx$   
 $-3py - 1py = -4py$   
and  $+r - q = +r - q$ .

3. From  $6a + 2b - (3a + b)$  subtract  $2a + 4b - (4a - b)$ .

$$6a + 2b - 3a - b$$

$$2a + 4b - 4a + b$$


---

$$4a - 2b + a - 2b =$$

$$5a - 4b \text{ Ans.}$$

Here, in writing down  $-(3a + b)$   
and  $-(4a - b)$ , we take care to change  
the signs within the brackets, be-  
cause the sign before each bracket  
is  $-$ .

### *Examples, to be proved by Addition.*

4. From  $5a - 6b - 3c$  subtract  $2a + 3b - 7c$ .

$$\text{Ans. } 3a - 9b + 4c.$$

5. From  $x^2 - xy$  take  $xy - y^2$ ; and from  $a + b$  take  $a - b$ .

$$\text{Ans. } x^2 - 2xy + y^2; \text{ and } 2b.$$

6. From  $5n^2 + n - 3$  take  $4n^2 - 3n - 2$ .

$$\text{Ans. } n^2 + 4n - 1.$$

7. From  $ay^3 - 7a^2y - a - c$  take  $3ay^2 - 2a^2y + a - b$ .

$$\text{Ans. } -2ay^2 - 5a^2y - 2a + b - c.$$

8. What is the difference between  $a^3 - a^2b + 2ab^2 - b^3$   
and  $2a^2b - ab^2$ .

$$\text{Ans. } a^3 - 3a^2b + 3ab^2 - b^3.$$

## MULTIPLICATION.

- VI. Multiply  $3 + 2$  by  $4$ .

$3 + 2 = 5$ , and  $5 \times 4 = 20 =$  product required;  
or thus,  $(3 + 2) \times 4 = 3 \times 4 + 2 \times 4 = 12 + 8 = 20 =$   
product required.

Multiply  $a + b$  by  $c$ , and also by  $d$ , and add the two results.

$$\begin{aligned}(a + b) \times c &= ac + bc \\ (a + b) \times d &= ad + bd\end{aligned}$$

---


$$ac + bc + ad + bd. \text{ Ans.}$$

This is obviously the same thing as multiplying  $a + b$  by  $c + d$ ; thus

$$\begin{array}{r} a + b \\ c + d \\ \hline ac + bc \phantom{+ ad + bd} \\ \phantom{ac + } + ad + bd \\ \hline ac + bc + ad + bd \end{array}$$

where  $a + b$  is multiplied by  $c$  and then by  $d$ , and the two results added together.

Multiply  $5 - 3$  by  $6$ .

$5 - 3 = 2$ , and  $2 \times 6 = 12 =$  product required;

or thus,  $(5 - 3) \times 6 = 30 - 18 = 12 =$  product required.

Multiply  $a - b$  by  $c$ , and also by  $d$ , and subtract the latter result from the former.

$$\begin{aligned}(a - b) \times c &= ac - bc \\ (a - b) \times d &= ad - bd\end{aligned}$$

---


$$ac - bc - ad + bd.$$

This is the same thing as multiplying  $a - b$  by  $c - d$ ; thus

$$\begin{array}{r} a - b \\ c - d \\ \hline ac - bc \phantom{- ad + bd} \\ \phantom{ac - } - ad + bd \\ \hline ac - bc - ad + bd \end{array}$$

where  $a - b$  is multiplied by  $c$ , and then by  $-d$ , and the two results collected into one.

On inspecting this example, it will readily appear that  $+$  multiplied by  $+$  gives  $+$  in the product;  $-$  multiplied by  $-$  gives  $+$ ;  $+$  multiplied by  $-$  gives  $-$ ; and  $-$  multiplied

by + gives -; or, in other words, *like* signs give +, and *unlike* signs give -.

Multiply  $2x$  by 3.

$$2x \times 3 = \text{three times } 2x = 2x + 2x + 2x = 6x,$$

$$\text{or thus, } 2 \times x \times 3 = 6x.$$

This result might have been obtained by multiplying the coefficient 2 by the multiplier 3, and writing the  $x$  after the product.

Multiply  $5a$  by  $2b$ .

$$5 \times a \times 2 \times b = 5 \times 2 \times a \times b = 10ab.$$

This result might have been obtained by *multiplying the coefficients* 5 and 2 together, and writing the quantities  $a$  and  $b$  after the product.

Multiply  $a^2$  by  $a^3$ .

$$a^2 = a \times a, \text{ and } a^3 = a \times a \times a;$$

$$\therefore a^2 \times a^3 = a \times a \times a \times a \times a = a^5;$$

$$\text{or thus, } a^2 \times a^3 = a^{2+3} = a^5.$$

This result might have been obtained by writing the quantity  $a$  with the *sum of the indices*, 2 and 3, for its index.

When no index is expressed, the index 1 is understood; thus,  $a = a^1$ .

From these considerations we derive the following

### *General Rules for Multiplication.*

If the signs are *alike*, write +; if *unlike*, -. Multiply the coefficients. Add the indices of the same letter.

### *Examples.*

1. Multiply  $3a^2 - 5ab + 2b^2$  by  $a^2 - 7ab$ .

$$\begin{array}{r} 3a^2 - 5ab + 2b^2 \\ a^2 - 7ab \\ \hline 3a^4 - 5a^3b + 2a^2b^2 \\ - 21a^3b + 35a^2b^2 - 14ab^3 \\ \hline 3a^4 - 26a^3b + 37a^2b^2 - 14ab^3 \end{array}$$

2. Multiply  $a^n - a^{n-1}b + a^{n-2}b^2$  by  $a - b$ .

$$\begin{array}{r}
 a^n - a^{n-1}b^1 + a^{n-2}b^2 \\
 a^1 - b^1 \\
 \hline
 a^{n+1} - a^n b + a^{n-1} b^2 \\
 \quad - a^n b \qquad \qquad + a^{n-1} b^2 - a^{n-2} b^{2+1} \\
 \hline
 a^{n+1} - 2a^n b + a^{n-1} b^2 + a^{n-1} b^2 - a^{n-2} b^{2+1} \\
 \hline
 \end{array}$$

3. Multiply  $6a$  by  $4$ ;  $3a$  by  $5b$ ; and  $x^4$  by  $x^3$ .

Ans.  $24a$ ;  $15ab$ ; and  $x^7$ .

4. Multiply  $2x - 4y + z$  by  $3x$ ; and  $a^2 + 2ab - b^2$  by  $a^3b^2$ .  
Ans.  $6x^2 - 12xy + 3xz$ ; and  $a^5b^2 + 2a^4b^3 - a^3b^4$ .

5. Multiply  $x^3 - x^2y + xy^2 - y^3$  by  $x + y$ ; and  $a^2 + b^2 + c^2 + ab - ac + bc$  by  $a - b + c$ .  
Ans.  $x^4 - y^4$ ; and  $a^3 - b^3 + c^3 + 3abc$ .

6. Multiply  $5x^3 + 4x^2 + 3x + 2$  by  $5x^3 - 4x^2$ ; and  $1 - x + x^4 - x^5$  by  $1 + x + x^2 + x^3$ .  
Ans.  $25x^6 - x^4 - 2x^3 - 8x^2$ ; and  $1 - x^6$ .

7. Find the product of  $a + b$  and  $a + b$ ; of  $a - b$  and  $a - b$ ; and of  $a + b$  and  $a - b$ ; and learn the results by heart.

Ans.  $(a + b)(a + b) = a^2 + 2ab + b^2$ ;  $(a - b)(a - b) = a^2 - 2ab + b^2$ ; and  $(a + b)(a - b) = a^2 - b^2$ .

From this example it appears that

1. The *square of the sum* of two quantities is equal to the *sum of their squares, together with twice their product*.

2. The *square of the difference* of two quantities is equal to the *sum of their squares, diminished by twice their product*.

3. The *product of the sum and difference* of two quantities is equal to the *difference of their squares*.

The student is recommended to commit these three theorems to memory.

## DIVISION.

VII. Division is just the reverse of Multiplication; thus,

$$\therefore 2x \times 3 = 6x; \therefore 6x \div 3, \text{ or } \frac{6x}{3} = 2x;$$

10. Divide  $8x^4y + 2x^3y - 2x^2 - 8x^2y + x$  by  $4x^2y + 3xy - 1$ ; and  $x^4 - y^4$  by  $x - y$ .

Ans.  $2x^2 - x$ ; and  $x^3 - x^2y + xy^2 + y^3$ .

11. Divide  $18x^3 - 33x^2 + 44x - 35$  by  $6x - 7$ ; and  $1 - x$  by  $1 + x$ . Ans.  $3x^2 - 2x + 5$ ; and  $1 - 2x + 2x^2 - 2x^3 + \&c$ .

$$12. \frac{a^4 + 2a^2b^2 + b^4}{a^2 + b^2}; \frac{x^6 - 2x^3y^3 + y^6}{x^3 - y^3}; \frac{4a^3 - 1}{2a^4 + 1};$$

$$\text{and } \frac{a^m \cdot b^{n+2} - a^{m+1} \cdot b^{2n+1} + a^{m+2} \cdot b^{3n}}{a^{m-2}b^n}.$$

Ans.  $a^2 + b^2$ ;  $x^3 - y^3$ ;  $2a^4 - 1$ ; and  $a^2b^{n+2} - a^3b^{n+1} + a^4b^n$ .

### Miscellaneous Examples.

1. Find the numerical value of the following expressions, when  $x = 4$ ,  $y = 3$ , and  $z = 6$ .

$$(1.) x^2 + 4y - 3z. \quad (2.) \frac{2x + 6y}{18} - \frac{z}{y}.$$

$$(3.) \frac{x(z - y)}{3} + \frac{z}{x - y}.$$

2. Add together  $5a^2b - 2ab^2 - 3a^2b^2$ ,  $2a^2b^2 - 7a^2b + 5ab$ ; and  $ab^2 - a^2b$ .

3. Subtract  $2x^3 - 3x^2 + 8x - 5$  from  $5x^3 - x^2 - 2x + 1$ .

4. Multiply  $4a^3x^2 - 7a^2x - 3a$  by  $2ax^3 - a^2$ ; and  $a^2 + ab + b^2$  by  $a - b$ .

5. Divide  $x^4 + x^3y^2 + y^4$  by  $x^2 + xy + y^2$ ; and  $x^3 + y^3$  by  $x + y$ ; and prove each operation by multiplication.

6. Find the sum of  $a^3\sqrt{x} - b\sqrt{x}$ ,  $5a^3\sqrt{x} + 3b\sqrt{x}$ ,  $2b\sqrt{x} - 7a^3\sqrt{x}$ , and  $3a^3\sqrt{x} + 4b\sqrt{x}$ ; and from  $(3a - x)x$  take  $(x + 2x)x$ .

7. Separate into simple factors  $a^3 - x^2$ ;  $a^2 + 2ax + x^2$ ;  $x^4 - 1$ ;  $4a^4 - 9b^2$ ;  $x^2 - 4bx + 4b^2$ ; and  $a^{2m} - b^{2n}$ .

8. Divide  $ax^3 - (a^3 + b)x^2 + b^2$  by  $ax - b$ ; and multiply  $px^3 + qx - r$  by  $mx - n$ .

9. Express in simple factors  $a^2a$ ;  $mn - m$ ;  $ab^2 - a$ ;  $-bx^2 + bcx^2 - b^2x^2$ ; and  $-c^2x - bx - a$ .

10. Express without brackets  $mn (m + n)$ ;  $xy (x - y)$ ;  $(p + r) \cdot (p - r)$ ;  $-(r + av)x$ ; and  $-\{x^2 + (x - 1)\}y$ .

## SIMPLE EQUATIONS.

VIII. When two quantities, or combinations of quantities, are equal, they form an equation; thus  $4 \times 3 = 7 + 5$  is an equation. Also  $10x = 60$  is an equation, in which the value of  $x$  is evidently 6, since  $10 \times 6 = 60$ ; and, if both sides of the equation be divided by 10, we shall have  $x = 6$  at once. Again,  $x + 5 = 15$  is an equation, from each side of which if we subtract 5, we shall have  $x = 10$ . Also  $x - 4 = 20$  is an equation, to each side of which if we add 4, we shall have  $x = 24$ . And  $\frac{1}{4}x = 2$  is an equation in which the value of  $x$  is 8, since the fourth part of 8 is 2; and, if both sides of this equation be multiplied by 4, we shall at once have  $x = 8$ .

Let  $x + 2a = 8a - 2x$  be a given equation.

Add  $2x$  to both sides, then

$$x + 2x + 2a = 8a.$$

Subtract  $2a$  from both sides, then

$$x + 2x = 8a - 2a.$$

On comparing this with the original equation, it appears that the  $2x$  and the  $2a$  have each been *transposed*, or removed from one side of the equation to the other, and have had their *signs changed*.

Now  $\therefore x + 2x = 3x$ , and  $8a - 2a = 6a$

$$\therefore 3x = 6a$$

and taking a third part of each member of this equation, we have

$x = 2a$  for the value of the previously unknown quantity  $x$ .

Hence it appears that we may transpose any of the terms of an equation if we change their signs, and that whatever we do to one member of an equation, we must also do to the other, in order to preserve the equality.

Solve the following equations:

1.  $x + 3 = 18 - 4x$ .

Ans.  $x = 3$ .

2.  $x + 3a = 18a - 4x$ .

Ans.  $x = 3a$ .

$$3. 4x - 2a = 3x + 2b. \quad \text{Ans. } x = 2(a + b).$$

$$4. 7 + 6x - 4 = 12 + 3x. \quad \text{Ans. } x = 3.$$

$$5. 4(x - 2) = 10x - 38. \quad \text{Ans. } x = 5.$$

$$6. ax + c = a - bx. \quad \text{Ans. } x = \frac{a - c}{a + b}.$$

$$7. \frac{y}{12} - 8 = -6, \quad \text{Ans. } y = 24.$$

$$8. 5ax - 1 = 3a(x + b). \quad \text{Ans. } x = \frac{3ab + 1}{2a}.$$

*Problems.*

1. A man buys a cow and her calf for 21*l.*, and the cow is worth 6 times as much as the calf: what is the value of each?

Let  $x$  pounds represent the value of the calf;

then  $\therefore$  the cow is worth 6 times as much,

$\therefore 6x$  is the value of the cow.

But, by the question, the value of the cow + the value of the calf = 21 pounds.

$$\therefore 6x + x = 21$$

$$\text{that is, } 7x = 21$$

$$\therefore x = 3 \text{ pounds} = \text{value of calf,}$$

$$\text{and } 6x = 18 \text{ pounds} = \text{..... cow.}$$

2. A had 100*l.* and B 48*l.*; B gave a certain sum away, and A twice as much, and then A had three times as much as B had. What did A give away?

Let  $x$  = number of pounds B gave away,

then  $2x$  = ..... A .....

Now each man's whole money minus the money he gave away = the money he had left.

$$\therefore 100 - 2x = \text{money A had left,}$$

$$\text{and } 48 - x = \text{... B .....}$$

But, by the question, A had left 3 times as much as B,

$$\therefore 100 - 2x = 3(48 - x)$$

$$\text{that is, } 100 - 2x = 144 - 3x,$$

$$\text{By transposition, } 3x - 2x = 144 - 100;$$

$$\text{that is, } x = 44\text{i.} = \text{B's gift,}$$

$$\text{and } 2x = 88\text{i.} = \text{A's gift.}$$

3. Divide 600*l.* among three persons, A, B, and C, so that B may have twice as much as A, and C as much as A and B together.      Ans. A 100*l.*, B 200*l.*, and C 300*l.*

4. The difference of two numbers is 7, and if 8 times the less number be subtracted from 3 times the greater, the remainder will be 6: find the numbers.      Ans. 10 and 3.

5. A man and his son earn 96 shillings in a month: now the man's labour is 5 times as valuable as the son's: how much ought each to receive?      Ans. The man 4*l.*, the son 16*s.*

6. Two persons, A and B, invest equal sums of money in railway shares; A gains 300*l.* and B loses 450*l.*, after which A's money is 6 times as much as B's: what money did each invest?      Ans. 600*l.*

7. A line,  $a$  feet long, is to be divided into two such parts that one part may be  $b$  times the length of the other: find the length of each part.

Ans.  $\frac{a}{b+1}$  feet = less part,  $\frac{ab}{b+1}$  feet = greater.

Find the length of the parts when  $a = 20$  and  $b = 4$ .

Ans. 4 and 16.

## CHAPTER II.

### FRACTIONS.

IX. Algebraic Fractions are the same in principle as Fractions in Common Arithmetic, consequently the same rules apply to both. Suppose any unit, as an orange, to be divided into  $b$  equal parts, and  $a$  of those parts to be taken, we shall then have the fraction  $\frac{a}{b}$ , just as, when we divide the unit into

8 equal parts and take 5 of them, we have the fraction  $\frac{5}{8}$ .

Since  $\frac{a}{b}$  expresses the *quotient* of  $a$  by  $b$ , and since quotient  $\times$  divisor = dividend

$$\therefore \frac{a}{b} \times b = a;$$

that is, if we *multiply a fraction by its denominator*, we obtain its *numerator*.



Multiply each member of the equation by  $n$ , then

$$nb \times \frac{a}{b} = na$$

$$\therefore \frac{a}{b} = \frac{na}{nb};$$

that is, the value of a fraction is not altered by multiplying both numerator and denominator by the same quantity.

*Cor.*  $\therefore \frac{na}{nb} = \frac{a}{b}$ ,  $\therefore$  the value of a fraction is not altered by dividing both numerator and denominator by the same quantity.

### Examples.

1. Multiply  $\frac{x}{2}$  by 2;  $\frac{x}{3}$  by 12;  $\frac{2y}{5}$  by 15; and  $\frac{3y}{4}$  by  $5x$ .

$\frac{x}{2} \times 2 = x$ ,  $\therefore$  twice the half of anything = the whole of that thing.

$\frac{x}{2} \times 12 = 4x$ ,  $\therefore$  12 times the third of anything = 12 thirds = 4 times the whole of that thing. Hence, when we have to multiply a fraction by a number, we may, if we can, previously divide the multiplier by the denominator.

$\frac{2y}{5} \times 15 = 6y$ ,  $\therefore$  15 times  $\frac{2}{5} = \frac{30}{5} = 6$ , or  $\therefore 15 \div 5 = 3$ , and  $2y \times 3 = 6y$ .

$\frac{3y}{4} \times 5x = \frac{15xy}{4}$   $\therefore$  5 times  $\frac{3}{4} = \frac{15}{4}$ , and  $y \times x = xy$ .

2. Multiply  $\frac{x}{3} + \frac{3x}{4} - \frac{5x}{6} + \frac{3x}{8} - \frac{7x}{12}$  by 24, and add the results. Ans.  $8x + 18x - 20x + 9x - 14x = x$ .

3. Multiply  $\frac{3y}{2} - \frac{2y}{5} - \frac{y}{4} + \frac{7y}{10} - \frac{5y}{4}$  by 20, and add the results. Ans.  $6y$ .

4. Multiply  $\frac{4}{x} - \frac{7}{2x} + \frac{3}{8x} - \frac{9}{4x} + \frac{3}{2}$  by  $8x$ , and add the results. Ans.  $12x - 11$ .

5. Multiply  $\frac{x}{5} + \frac{5}{x} - \frac{x}{10} - \frac{2}{x^2} + 3$  by  $10x^2$ .

Ans.  $x^3 + 30x^2 + 50x - 20$ .

These examples will aid the student in clearing equations from fractions.

### GREATEST COMMON MEASURE.

X. A *measure* of a quantity is any factor which will divide it without remainder: thus 2 is a measure of  $2a$ ;  $ax$  is a measure of  $ax^2$ ;  $a + b$  is a measure of  $(a + b)a$ .

A *common measure* of two or more quantities is any factor which will divide them all without remainder; and their *greatest common measure* is the greatest factor which will so divide them: thus  $a$  is a common measure of  $2a$ ,  $6a^2x$ , and  $4ax^2$ , and their greatest common measure is  $2a$ : also  $3(a + b)$  is the greatest common measure of  $6(a + b)^2$ ,  $9(a^2 - b^2)$  and  $12(a^3 - b^3)$ , for  $6(a + b)^2 = 6(a + b) \cdot (a + b)$ ,  $9(a^2 - b^2) = 9(a + b)(a - b)$ , and  $12(a^3 - b^3) = 12(a + b)(a^2 - ab + b^2)$ ; where  $3(a + b)$  is evidently the greatest common factor.

This mode of proceeding, by separating the quantities into their simple factors, will frequently enable the student to ascertain the greatest common measure.

#### Examples.

1. Find the g. c. m. of  $x^2 + 2x - 3$  and  $x^2 + 5x + 6$ .

$$\begin{aligned} x^2 + 2x - 3 &= x^2 + 3x - x - 3 = (x + 3)x - (x + 3) \\ &= (x + 3)(x - 1) \end{aligned}$$

$$\begin{aligned} x^2 + 5x + 6 &= x^2 + 3x + 2x + 6 = x(x + 3) + 2(x + 3) \\ &= (x + 3)(x + 2) \end{aligned}$$

$\therefore x + 3$  is the g. c. m.

If  $x^2 + 2x - 3$  and  $x^2 + 5x + 6$  were the two terms of a fraction, it could be reduced to its lowest terms by dividing both numerator and denominator by the greatest common measure  $x + 3$ ; thus

$$\frac{x^2 + 2x - 3}{x^2 + 5x + 6} = \frac{(x + 3)(x - 1)}{(x + 3)(x + 2)} = \frac{x - 1}{x + 2}, \text{ the fraction in}$$

its lowest terms.

2. Find the G. C. M. of  $\frac{x^3 - 6x^2 + 11x - 6}{x^3 + 4x^2 + x - 6}$ , and reduce the fraction to its lowest terms.

$$\begin{aligned}\frac{x^3 - 6x^2 + 11x - 6}{x^3 + 4x^2 + x - 6} &= \frac{x^3 - x^2 - 5x^2 + 5x + 6x - 6}{x^3 - x^2 + 5x^2 - 5x + 6x - 6} \\ &= \frac{x^2(x-1) - 5x(x-1) + 6(x-1)}{x^2(x-1) + 5x(x-1) + 6(x-1)} \\ &= \frac{(x-1)(x^2 - 5x + 6)}{(x-1)(x^2 + 5x + 6)} = \frac{x^2 - 5x + 6}{x^2 + 5x + 6} \\ &= \text{the fraction in its lowest terms, and } x-1 = \text{G. C. M.}\end{aligned}$$

$$\begin{aligned}3. \frac{x^3 - 9x + 20}{x^3 + 6x - 55} &= \frac{x^3 - 4x - 5x + 20}{x^3 - 5x + 11x - 55} \\ &= \frac{x(x-4) - 5(x-4)}{x(x-5) + 11(x-5)} = \frac{(x-4)(x-5)}{(x-5)(x+11)} \\ &= \frac{x-4}{x+11} = \text{the fraction in its lowest terms,}\end{aligned}$$

and  $x-5 = \text{G. C. M.}$

$$\begin{aligned}4. \frac{x^5 + x^3y^2 + x^2y + y^3}{x^4 - y^4} &= \frac{x^3(x^2 + y^2) + y(x^2 + y^2)}{(x^2 + y^2)(x^2 - y^2)} \\ &= \frac{x^3 + y}{x^2 - y^2} = \text{the fraction in its lowest terms,}\end{aligned}$$

and  $x^2 + y^2 = \text{G. C. M.}$

$$\begin{aligned}5. \frac{ab + 2a^2 - 3b^2 - 4bc - ac - c^2}{9ac + 2a^2 - 5ab + 4c^2 + 8bc - 12b^2} \\ &= \frac{2a^2 + ac + 3ab - 2ab - bc - 3b^2 - 2ac - c^2 - 8bc}{2a^2 - 8ab + 8ac + ac - 4bc + 4c^2 + 3ab - 12b^2 + 12bc} \\ &= \frac{a(2a + c + 3b) - b(2a + c + 3b) - c(2a + c + 3b)}{2a(a - 4b + 4c) + c(a - 4b + 4c) + 3b(a - 4b + 4c)} \\ &= \frac{(a-b-c)(2a+c+3b)}{(2a+c+3b)(a-4b+4c)} = \frac{a-b-c}{a-4b+4c} \\ &= \text{the fraction in its lowest terms,} \\ &\text{and } 2a + c + 3b = \text{G. C. M.}\end{aligned}$$

XI. A quantity is a *multiple* of another when it contains that other a certain number of times without remainder: thus  $6a$  is a multiple of  $2a$ ;  $na$  is a multiple of  $a$ .

If a quantity measures another, it will also measure any multiple of that other. Thus, suppose  $b$  measures  $a$  by the units in  $m$ , that is, let  $b$  be contained  $m$  times in  $a$ , then  $a = mb$ , and let  $na$  be a multiple of  $a$ , then  $na = nmb$ ; hence  $b$  measures  $na$  by the units in  $nm$ .

If a quantity measures two others, it will also measure their sum and difference. Thus, let  $x$  measure  $a$  by the units in  $m$ , and  $b$  by the units in  $n$ , then  $a = mx$ , and  $b = nx$ ,  $\therefore a \pm b = mx \pm nx = (m \pm n)x$ : hence  $x$  measures their sum  $(a + b)$  by the units in  $(m + n)$ , and their difference  $(a - b)$  by the units in  $(m - n)$ .

We can now investigate a general rule for finding the greatest common measure of any two numbers.

Let the numbers be represented by  $a$  and  $b$ ,  $a$  being  $> b$ .

Suppose  $b$  is contained in  $a$ ,  $p$  times with remainder  $c$ , then  $a = pb + c$ ;

Suppose  $c$  is contained in  $b$ ,  $q$  times with remainder  $d$ , then  $b = qc + d$ ;

Suppose  $d$  is contained in  $c$ ,  $r$  times with remainder 0, then  $c = rd$ .

$$\begin{array}{r}
 b) a (p \\
 \underline{pb} \\
 c) b (q \\
 \underline{qc} \\
 d) c (r \\
 \underline{dr} \\
 0
 \end{array}$$

Now  $\therefore d$  measures  $c$ ,  $\therefore d$  measures  $qc$ , and  $qc + d$ , that is,  $b$ ;

$\therefore d$  measures  $pb$  and  $pb + c$ , that is  $a$ .

Hence the last divisor  $d$  measures both the given numbers  $a$  and  $b$ .

$d$  is the greatest common measure of  $a$  and  $b$ : for suppose  $\delta$  to be the greatest common measure, then

5.  $2x^2 - xy - 6y^2$ , and  $3x^2 - 8xy + 4y^2$ . Ans.  $x - 2y$ .

6.  $2x^3 - 3x^2 - 2x + 3$ , and  $3x^4 + 2x^3 - 2x^2 - 2x - 1$ .  
Ans.  $x^3 - 1$ .

Reduce to the lowest terms

7.  $\frac{x^2 + 2ax + a^2}{x^2 - a^2}$  and  $\frac{x^3 - 2xy + y^3}{x^3 - y^3}$ . Ans.  $\frac{x+a}{x-a}$  and  $\frac{x-y}{x+y}$ .

8.  $\frac{a^3 + b^3}{(a+b)^2}$  and  $\frac{(a+b)^4}{a^2 - b^2}$ . Ans.  $\frac{a^2 - ab + b^2}{a+b}$  and  $\frac{(a+b)^2}{a-b}$ .

9.  $\frac{x^3 + x - 2}{2x^3 - 3x + 1}$  and  $\frac{x^{3m} + x^{2m} - 2}{x^{2m} + x^m - 2}$ .  
Ans.  $\frac{x+2}{2x-1}$  and  $\frac{x^{2m} + 2x^m + 2}{x^m + 2}$ .

10.  $\frac{x^3 - x^2y - xy^2 + y^3}{x^4 - y^4}$  and  $\frac{3x^3 - 22x - 15}{5x^4 - 17x^3 + 18x}$ .  
Ans.  $\frac{x-y}{x^2 + y^2}$  and  $\frac{3x^3 + 9x + 5}{5x^3 - 2x^2 - 6x}$ .

# ADDITION, SUBTRACTION, MULTIPLICATION, AND DIVISION OF ALGEBRAIC FRACTIONS.

## Examples.

1. Add together  $\frac{2x-5}{3}$  and  $\frac{x-1}{2x}$ .

Here the least common multiple of the denominators is  $6x$ , and the two fractions can be reduced so as to have a common denominator by multiplying both terms of the first by  $2x$ , and both terms of the second by  $3$ ; thus,

$$\left. \begin{array}{l} \frac{2x-5}{3} \times \frac{2x}{2x} = \frac{4x^2-10x}{6x} \\ \frac{x-1}{2x} \times \frac{3}{3} = \frac{3x-3}{6x} \end{array} \right\} \therefore \frac{4x^2-7x-3}{6x} = \text{the sum.}$$

2. Find the sum of  $\frac{1}{2}a - \frac{1}{3}b + \frac{1}{5}c$ ,  $\frac{1}{4}a - \frac{1}{5}b - \frac{1}{9}c$  and

$\frac{1}{3}a + \frac{1}{4}b + \frac{1}{2}c$ .

$$\frac{1}{2}a - \frac{1}{8}b + \frac{1}{8}c = \frac{6}{12}a - \frac{20}{60}b + \frac{6}{30}c$$

$$\frac{1}{4}a - \frac{1}{8}b - \frac{1}{8}c = \frac{3}{12}a - \frac{12}{60}b - \frac{10}{30}c$$

$$\frac{1}{8}a + \frac{1}{4}b + \frac{1}{2}c = \frac{4}{12}a + \frac{15}{60}b + \frac{15}{30}c$$

$$\therefore \frac{13}{12}a - \frac{17}{60}b + \frac{11}{80}c = \text{sum required.}$$

3. Of  $\frac{5x^2}{2} - \frac{7y^2}{3}$ ,  $-\frac{4x^2}{3} + \frac{5y^2}{2}$  and  $\frac{7x^2}{4} - \frac{3y^2}{2}$ .

$$\frac{5x^2}{2} - \frac{7y^2}{3} = \frac{30x^2}{12} - \frac{14y^2}{6}$$

$$-\frac{4x^2}{3} + \frac{5y^2}{2} = -\frac{16x^2}{12} + \frac{15y^2}{6}$$

$$\frac{7x^2}{4} - \frac{3y^2}{2} = \frac{21x^2}{12} - \frac{9y^2}{6}$$

$$\therefore \frac{35}{12}x^2 - \frac{8}{6}y^2 = 2\frac{1}{2}x^2 - 1\frac{1}{3}y^2 = \text{sum.}$$

4. Collect  $\frac{3}{4(1-x)^2} + \frac{3}{8(1-x)} + \frac{1}{8(1+x)} - \frac{1-x}{4(1+x)^2}$  into a single fraction.

Taking the second and third, and reducing them to a common denominator,

$$\left. \begin{aligned} \frac{3}{8(1-x)} \times \frac{1+x}{1+x} &= \frac{3+3x}{8(1-x^2)} \\ \frac{1}{8(1+x)} \times \frac{1-x}{1-x} &= \frac{1-x}{8(1-x^2)} \end{aligned} \right\} = \frac{4+2x}{8(1-x^2)} = \frac{2+x}{4(1-x^2)} \text{ by addition.}$$

Taking the fraction just found and the fourth, and reducing them to a common denominator,

$$\left. \begin{aligned} \frac{2+x}{4(1-x^2)} \times \frac{1+x^2}{1+x^2} &= \frac{x^3+2x^2+x+2}{4(1-x^4)} \\ \frac{1-x}{4(1+x^2)} \times \frac{1-x^2}{1-x^2} &= \frac{1-x-x^2+x^3}{4(1-x^4)} \end{aligned} \right\} = \frac{3x^2+2x+1}{4(1-x^4)} \text{ by subtraction.}$$

Taking the fraction last found and the first, and reducing them to a common denominator, first finding the least common multiple,

$$\begin{aligned} & \frac{4(1-x) \quad 4(1-x)^3 \quad 4(1-x^2) \cdot (1+x^2)}{(1-x) \quad (1+x) \quad (1+x^2)} = 4(1-x)(1-x^4) = \text{I.C.M.} \\ & \left. \begin{aligned} \frac{3}{4(1-x)^3} \times \frac{1+x+x^2+x^3}{1+x+x^2+x^3} &= \frac{3+3x+3x^2+3x^3}{4(1-x)(1-x^4)} \\ \frac{1+2x+3x^3}{4(1-x^4)} \times \frac{1-x}{1-x} &= \frac{1+x+x^3-3x^3}{4(1-x)(1-x^4)} \end{aligned} \right\} = \\ & \frac{4+4x+4x^2}{4(1-x)(1-x^4)} = \frac{1+x+x^2}{1-x-x^4+x^5} \text{ by addition.} \end{aligned}$$

5. Find the difference between  $a+b$  and  $\frac{a}{2} - \frac{b}{2}$ .

$$\begin{aligned} a+b &= \frac{2a}{2} + \frac{2b}{2} \\ & \quad \frac{a}{2} - \frac{b}{2} \end{aligned}$$

$$\therefore \frac{a}{2} + \frac{3b}{2} = \text{difference required.}$$

6. Find the difference between  $\frac{a}{3} + \frac{b}{2} - \frac{c}{4}$  and  $\frac{a+b}{4} - \frac{c}{8}$ .

$$\begin{aligned} \frac{a}{3} + \frac{b}{2} - \frac{c}{4} &= \frac{4a}{12} + \frac{6b}{12} - \frac{2c}{8} \\ \frac{a+b}{4} - \frac{c}{8} &= \frac{3a}{12} + \frac{3b}{12} - \frac{c}{8} \end{aligned}$$

$$\frac{a}{12} + \frac{3b}{12} - \frac{c}{8} = \frac{a+3b}{12} - \frac{c}{8} \quad \text{Ans.}$$

7. Find the sum and difference of  $\frac{x}{x-2}$  and  $\frac{x}{x-3}$ .

$$\frac{x}{x-2} \times \frac{x-3}{x-3} = \frac{x^2-3x}{x^2-5x+6}$$

$$\frac{x}{x-3} \times \frac{x-2}{x-2} = \frac{x^2-2x}{x^2-5x+6}$$

$\therefore$  by addition  $\frac{2x^2-5x}{x^2-5x+6}$  = the sum,

and by subtraction  $\frac{-x}{x^2-5x+6}$  = the difference.

8. Multiply  $\frac{4x^2-1}{8}$  by  $\frac{2x+1}{2x-1}$ ; and  $\frac{2a^2}{a^2-b^2}$  by  $\frac{(a+b)^2}{4b^2a^2}$ .

$$\begin{aligned} \frac{4x^2-1}{8} \times \frac{2x+1}{2x-1} &= \frac{(2x+1)(2x-1)}{8} \times \frac{2x+1}{2x-1} \\ &= \frac{(2x+1)(2x+1)}{8} = \frac{4x^2+4x+1}{8}. \end{aligned}$$

$$\begin{aligned} \frac{2a^2}{a^2-b^2} \times \frac{(a+b)^2}{4b^2a^2} &= \frac{2a^2}{(a+b)(a-b)} \times \frac{(a+b)(a+b)}{2b^2 \times 2a^2} \\ &= \frac{a+b}{2b^2(a-b)}. \end{aligned}$$

9. Divide  $\frac{a+1}{b}$  by  $\frac{1+\frac{1}{a}}{b}$ ; and  $\left(\frac{1}{a} + \frac{1}{b}\right)$  by  $\left(\frac{1}{a} - \frac{1}{b}\right)$ .

$$\frac{a+1}{b} \div \frac{1+\frac{1}{a}}{b} = \frac{a+1}{b} \div \frac{a+1}{ab} = \frac{a+1}{b} \times \frac{ab}{a+1} = a.$$

$$\begin{aligned} \left(\frac{1}{a} + \frac{1}{b}\right) \div \left(\frac{1}{a} - \frac{1}{b}\right) &= \frac{b+a}{ab} \div \frac{b-a}{ab} = \\ \frac{b+a}{ab} \times \frac{ab}{b-a} &= \frac{b+a}{b-a}. \end{aligned}$$

\* This is obtained by multiplying both terms of the divisor by  $a$ .



10. Multiply  $\frac{x^2 - (a+b)x + ab}{x^2 - (a-b)x - ab}$  by  $\frac{x+b}{x-b}$ .

$$\begin{aligned}\frac{x^2 - ax - bx + ab}{x^2 - ax + bx - ab} &= \frac{x^2 - bx - ax + ab}{x^2 + bx - ax - ab} = \\ \frac{x(x-b) - a(x-b)}{x(x+b) - a(x+b)} &= \frac{(x-a)(x-b)}{(x-a)(x+b)} = \frac{x-b}{x+b} \\ \text{and } \frac{x-b}{x+b} \times \frac{x+b}{x-b} &= 1.\end{aligned}$$

11. Collect into one expression  $\frac{a^m}{(a+b)^n} + \frac{a^{m-2}b^3}{(a+b)^{n-1}} - \frac{a^{m-3}b^3}{(a+b)^{n-2}}$ .

Here  $\therefore (a+b)^n$  is the L. C. M. of the denominators, we take the second and third fractions, and reduce them to the denominator  $(a+b)^n$ ; thus

$$\left. \begin{aligned}\frac{a^{m-2}b^3}{(a+b)^{n-1}} \times \frac{a+b}{a+b} &= \frac{a^{m-1}b^3 + a^{m-2}b^3}{(a+b)^n} \\ \frac{a^{m-3}b^3}{(a+b)^{n-2}} \times \frac{(a+b)^2}{(a+b)^2} &= \frac{a^{m-1}b^3 + 2a^{m-2}b^3 + a^{m-3}b^4}{(a+b)^n}\end{aligned}\right\}$$

$$= \frac{-a^{m-2}b^3 - a^{m-3}b^4}{(a+b)^n} \text{ by subtraction,}$$

and prefixing the first fraction, we have

$$\frac{a^m - (a^{m-2} + a^{m-3}b)b^3}{(a+b)^n} = \text{the expression required.}$$

### Examples in Addition of Fractions.

1. Add together  $\frac{x}{2}$ ,  $\frac{2x}{9}$  and  $\frac{3x}{4}$ . Ans.  $\frac{23x}{18} = x + \frac{11}{18}x$ .

2.  $\frac{2x}{5} + \frac{3x}{2} + \frac{7x}{10}$ ; and  $\frac{x}{a+x} + \frac{a}{a-x}$ .

$$\text{Ans. } \frac{13x}{5}; \text{ and } \frac{a^2 + 2ax - x^2}{a^2 - x^2}.$$

$$3. \frac{3a^2}{2b} + \frac{2a}{5} + \frac{3b}{7a}; \text{ and } \frac{x}{x+3} + \frac{x}{x-3}.$$

$$\text{Ans. } \frac{105a^3 + 28a^2b + 30b^2}{70ab}; \text{ and } \frac{2x^2}{x^2-9}.$$

$$4. \frac{a+b}{a-b} + \frac{a-b}{a+b}; \text{ and } \frac{2x}{1-x^2} + \frac{1}{x+1}.$$

$$\text{Ans. } \frac{2(a^2+b^2)}{a^2-b^2}; \text{ and } \frac{1}{1-x}.$$

$$5. \frac{a^2+ab+b^2}{a+b} + \frac{b^2}{a-b}; \text{ and } \frac{x-1}{x^2+x+1} + \frac{1}{x-1}.$$

$$\text{Ans. } \frac{a(a^2+b^2)}{a^2-b^2}; \text{ and } \frac{2x^2-x+2}{x^2-1}.$$

$$6. \frac{x}{x^2-y^2} + \frac{y}{x+y} + \frac{1}{x-y}; \text{ and } \frac{a}{1-a} + \frac{a}{1+a} + \frac{1}{1+a}.$$

$$\text{Ans. } \frac{2x+xy-y^2+y}{x^2-y^2}; \text{ and } \frac{1}{1-a}.$$

*Examples in Subtraction of Fractions.*

$$1. \text{ From } \frac{5a}{2} \text{ take } \frac{a}{9}; \text{ and from } \frac{3(x+y)}{4} \text{ take } \frac{x-y}{8}.$$

$$\text{Ans. } \frac{13}{6}a; \text{ and } \frac{5x+7y}{8}.$$

$$2. \frac{5y+2}{7} - \frac{2y+1}{3}; \text{ and } \frac{5x-3}{x+1} - \frac{3x+2}{x-1}.$$

$$\text{Ans. } \frac{y-1}{21}; \text{ and } \frac{2x^2-13x+2}{x^2-1}.$$

$$3. \frac{a+2b}{a-2b} - \frac{a-2b}{a+2b}; \text{ and } \frac{1}{x-y} - \frac{1}{x+y}.$$

$$\text{Ans. } \frac{8ab}{a^2-4b^2}; \text{ and } \frac{2y}{x^2-y^2}.$$

$$4. \frac{1}{y-z} - \frac{1}{y^2-z^2}; \text{ and } \frac{2x^2-2x+1}{x^2-x} - \frac{x}{x-1}.$$

$$\text{Ans. } \frac{y+z-1}{y^2-z^2}; \text{ and } 1 - \frac{1}{x}.$$

$$5. \frac{x^3-x+1}{x-1} - \frac{2}{x+1}; \text{ and } \frac{a}{(1-a)^2} - \frac{a^3}{(1-a)^3} + \frac{1}{1-a}.$$

$$\text{Ans. } \frac{x^3-2x+3}{x^2-1}; \text{ and } \frac{1-a-a^3}{(1-a)^3}.$$

$$6. \frac{x^2+y^2}{x^2-y^2} + \frac{x}{x+y} - \frac{y}{x-y}; \text{ and } 1 - \frac{1}{(x-y)^2} - \frac{1}{x^2-y^2}.$$

$$\text{Ans. } \frac{2x}{x+y}; \text{ and } 1 - \frac{2y}{(x+y)(x-y)^2}.$$

$$7. \frac{1}{2} \cdot \frac{3m+2n}{3m-2n} - \frac{1}{2} \cdot \frac{3m-2n}{3m+2n}. \quad \text{Ans. } \frac{6mn}{9m^2-4n^2}.$$

$$8. \frac{x^{2n}}{x^n-1} - \frac{x^{2n}}{x^n+1} - \frac{1}{x^n-1} + \frac{1}{x^n+1}. \quad \text{Ans. } x^{2n} + 2.$$

*Examples in Multiplication of Fractions.*

$$1. \text{ Multiply } \frac{3a}{5} \text{ by } \frac{a}{4}; \text{ and } \frac{2x}{3} \text{ by } \frac{xy^2}{6}.$$

$$\text{Ans. } \frac{3a^2}{20}; \text{ and } \frac{x^2y^2}{9}.$$

$$2. \frac{2x}{x-y} \times \frac{x^2-y^2}{8}; \text{ and } \frac{x^2-1}{3} \times \frac{6a}{x+1}.$$

$$\text{Ans. } \frac{x^2+xy}{4}; \text{ and } 2a(x-1).$$

$$3. \frac{a^2-b^2}{a} \times \frac{1}{a+b} \times \frac{a}{a-b}; \text{ and } \frac{15z-30}{2z} \times \frac{3z^2}{5z-10}.$$

$$\text{Ans. } 1; \text{ and } \frac{9}{2}z.$$

$$4. \frac{(x-1)^2}{y^3} \times \frac{(x+1)y^3}{x-1}; \text{ and } \left(m + \frac{1}{m} - 1\right) \times$$

$$\left(m + \frac{1}{m} + 1\right). \quad \text{Ans. } \frac{x^2-1}{y}; \text{ and } m^2+1 + \frac{1}{m^2}.$$

$$5. \left(x - \frac{y^2}{x}\right) \left(\frac{x}{y} + \frac{y}{x}\right); \text{ and } \frac{4x^4-1}{9x^6-y^2} \cdot \frac{3x^3+y}{2x^2-1}.$$

$$\text{Ans. } \frac{x^4-y^4}{x^2y}; \text{ and } \frac{2x^3+1}{3x^3-y}.$$

$$6. \left\{ \frac{a^4 x}{a^2 - x^2} - a^2 x - x^3 \right\} \times \frac{a+x}{x^3}. \quad \text{Ans. } \frac{x^3}{a-x}.$$

*Examples in Division of Fractions.*

$$1. \text{ Divide } \frac{5m^2}{n} \text{ by } \frac{10}{3n}; \text{ and } \frac{3x}{2x-2} \text{ by } \frac{2x}{x-1}.$$

$$\text{Ans. } \frac{3}{2}m^2; \text{ and } \frac{3}{4}.$$

$$2. \frac{4a+2}{3} \div \frac{2a+1}{5a}; \text{ and } \frac{(x+y)^2}{x-y} \div \frac{x+y}{(x-y)^2}.$$

$$\text{Ans. } \frac{10a}{3}; \text{ and } x^2 - y^2.$$

$$3. \left( x + \frac{x}{x-1} \right) \div \left( x - \frac{x}{x-1} \right); \text{ and } \left( x^4 - \frac{1}{x^3} \right) \div \left( x^2 + \frac{1}{x} \right).$$

$$\text{Ans. } \frac{x}{x-2}; \text{ and } x^2 - \frac{1}{x}.$$

$$4. \frac{x^3 - 3x^2a + 3xa^2 - a^3}{x+a} \div \frac{x-a}{x+a}. \quad \text{Ans. } (x-a)^2.$$

$$5. \frac{x^4 - y^4}{a^3 + b^3} \div \frac{x-y}{a^2 - ab + b^2}. \quad \text{Ans. } \frac{x^3 + x^2y + xy^2 + y^3}{a+b}.$$

$$6. \left( x^2 + 2 + \frac{1}{x^2} \right) \div \frac{x + \frac{1}{x}}{a}. \quad \text{Ans. } \frac{ax^2 + a}{x}.$$

Find the value of  $x$  in the following equations.

$$1. 3x + 4 = \frac{5x+4}{2} + 8. \quad \text{Multiply by 2.}$$

$$6x + 8 = 5x + 4 + 16. \quad \text{Transpose.}$$

$$6x - 5x = 4 + 16 - 8$$

$$\therefore x = 12.$$

$$2. \frac{30+x}{x} - 5 = \frac{6}{x}. \quad \text{Multiply by } x.$$

$$30 + x - 5x = 6. \quad \text{Transpose.}$$

$$x - 5x = 6 - 30$$

$$-4x = -24. \quad \text{Divide by } -4.$$

$$\therefore x = 6.$$

3.  $\frac{x}{2} - \frac{5x+4}{8} = 7 - \frac{8x-2}{8}$ . Multiply by 8, the L.C.M. of the denominators.

$$3x - 10x - 8 = 42 - 16x + 4, \text{ Transpose.}$$

$$3x - 10x + 16x = 42 + 4 + 8$$

$$9x = 54$$

$$\therefore x = 6.$$

4.  $\frac{2}{x+1} + \frac{5}{2x+2} + \frac{6x-6}{x^2-1} = 2\frac{5}{8}.$

Here  $\frac{6x-6}{x^2-1} = \frac{6(x-1)}{(x+1)(x-1)}$ , hence the equation becomes

$$\frac{2}{x+1} + \frac{5}{2(x+1)} + \frac{6}{x+1} = \frac{21}{8}. \text{ Multiply by } 8(x+1).$$

$$16 + 20 + 48 = 21(x+1)$$

$$84 = 21(x+1)$$

$$4 = x+1$$

$$\therefore 3 = x.$$

5.  $\frac{ax+b}{c} + \frac{ax+b}{cx+b} = \frac{2ax+d}{2c} + \frac{b}{c}$ . Multiply by  $2c$ .

$$2ax + 2b + \frac{2acx + 2bc}{cx+b} = 2ax + d + 2b.$$

$\therefore 2ax$  and  $2b$  are found on each side of the equation with the same sign, they may be omitted; and the equation becomes

$$\frac{2acx + 2bc}{cx+b} = d. \text{ Multiply by } cx+b.$$

$$2acx + 2bc = cdx + bd. \text{ Transpose.}$$

$$2acx - cdx = bd - 2bc$$

$$(2ac - cd)x = bd - 2bc$$

$$\therefore x = \frac{bd - 2bc}{2ac - cd}.$$

\* The sign — in front of these fractions changes the signs of their numerators; for it indicates that each fraction is to be *subtracted* from the quantity preceding it.

$$6. \frac{7x + \frac{13}{2}}{10} + \frac{11x - \frac{x-3}{2}}{12} = \frac{3x+1}{5} + \frac{43x - \frac{3-8x}{2}}{22}.$$

$$\text{Here } \frac{7x + \frac{13}{2}}{10} = \frac{14x + 13}{20}.$$

$$\frac{11x - \frac{x-3}{2}}{12} = \frac{11x - \frac{2x-3}{4}}{12} = \frac{44x - 2x + 3}{48} =$$

$$\frac{42x + 3}{48} = \frac{14x + 1}{16}.$$

$$\frac{43x - \frac{3-8x}{2}}{22} = \frac{86x - 3 + 8x}{44} = \frac{94x - 3}{44}.$$

Hence the equation becomes

$$\frac{14x + 13}{20} + \frac{14x + 1}{16} = \frac{3x + 1}{5} + \frac{94x - 3}{44}.$$

Multiply by 880, the l. c. m. of the denominators.

$$616x + 572 + 770x + 55 = 528x + 176 + 1880x - 60,$$

$$1386x - 2408x = 116 - 627$$

$$-1022x = -511$$

$$\therefore x = \frac{-511}{-1022} = \frac{1}{2}.$$

### Problems.

1. Divide 100 into two such parts that, if the one be divided by 15 and the other by 5, the sum of the quotients shall be 10.

Let  $x$  be one part, then  $100 - x$  will be the other.

Now  $\frac{x}{15}$  and  $\frac{100-x}{5}$  are the two quotients.

Hence, by the question,  $\frac{x}{15} + \frac{100-x}{5} = 10.$

$$\begin{aligned}x + 300 - 3x &= 150 \\- 2x &= 150 - 300 \\&= -150\end{aligned}$$

$\therefore x = 75 =$  one part,  
and  $100 - x = 100 - 75 = 25 =$  the other.

Or thus; let  $x$  and  $y$  be the parts,

$$\text{then } x + y = 100 \quad (1)$$

$$\text{and } \frac{x}{15} + \frac{y}{5} = 10 \quad (2). \text{ Multiply by 15.}$$

$$\begin{array}{rcl}x + 3y & = & 150 \\(1) \quad x + y & = & 100\end{array} \left. \vphantom{\begin{array}{rcl}x + 3y & = & 150 \\x + y & = & 100\end{array}} \right\} \text{ Subtract.}$$

$$2y = 50 \therefore y = 25 = \text{one part.}$$

$$(1) \quad x = 100 - y = 100 - 25 = 75 = \text{the other.}$$

2. A person in a foreign town wishes to exchange a sovereign for 25 pieces of the two kinds of coin used there; and he finds that 30 of the one, or 15 of the other, are equal to a sovereign. How many must he have of each?

Let  $x$  be the number of one kind,  
and  $y$  ..... the other;

$$\text{then } x + y = 25 \quad \text{.....} \quad (1)$$

$$\begin{array}{cccc} \text{coins.} & \text{coins.} & \text{shill.} & \text{shill.} \\ 30 & : & x & :: 20 : \frac{2x}{3} = \text{value of the } x \text{ coins.} \end{array}$$

$$15 : y :: 20 : \frac{4y}{3} = \text{..... } y \dots$$

$$\text{Hence, by the question } \frac{2x}{3} + \frac{4y}{3} = 20 \text{ shillings .....} \quad (2)$$

$$\text{Multiply (1) by 2, } \begin{array}{rcl} 2x + 4y & = & 60 \\ 2x + 2y & = & 50 \end{array} \left. \vphantom{\begin{array}{rcl} 2x + 4y & = & 60 \\ 2x + 2y & = & 50 \end{array}} \right\} \text{ Subtract}$$

$$2y = 10 \therefore y = 5$$

$$(1) \quad x = 25 - y = 25 - 5 = 20$$

Ans. 20 and 5.

3. A person lays by  $\frac{1}{4}$ th of what he spends. His income being diminished, while his expenditure remains the same, he finds he now lays by only half the former sum. What part of the original income was the diminution?

Let  $2x$  pounds be what he lays by,  
then  $8x$  ..... is ..... spends;

$\therefore 2x + 8x = 10x =$  his original income.

Again  $8x$  is still what he spends,  
 $x$  ..... lays by;

$\therefore 8x + x = 9x =$  his diminished income,

$\therefore 10x - 9x = x =$  the diminution,

Hence  $\frac{x}{10x} = \frac{1}{10} =$  Ans.

4. A performs a journey at a certain rate; had he travelled  $\frac{1}{2}$  a mile an hour quicker, he would have performed the journey in  $\frac{4}{5}$  of the time; but had he travelled  $\frac{1}{2}$  a mile slower, he would have been  $2\frac{1}{2}$  hours longer on the road. Find the distance and his rate.

Let  $x$  miles be his rate per hour,  
and  $y$  miles be the whole distance,

then  $\frac{y}{x}$  is the number of hours the journey occupies.

Now  $x + \frac{1}{2}$  or  $\frac{2x+1}{2}$  is the increased rate;

$\therefore y \div \frac{2x+1}{2}$  or  $\frac{2y}{2x+1}$  is the hours he would be on the road, by travelling  $\frac{1}{2}$  hour quicker.

Hence, by the question,  $\frac{2y}{2x+1} = \frac{4}{5} \cdot \frac{y}{x}$  ..... (1)

Again  $x - \frac{1}{2}$  or  $\frac{2x-1}{2}$  is the diminished rate;

$\therefore y \div \frac{2x-1}{2}$  or  $\frac{2y}{2x-1}$  is the hours he would be on the road, by travelling  $\frac{1}{2}$  hour slower.



Hence, by the question,  $\frac{2y}{2x-1} = \frac{y}{x} + 2\frac{1}{2}$  ..... (2)

Divide (1) by  $2y$ ,  $\frac{1}{2x+1} = \frac{2}{5x}$

$$\begin{aligned} 5x &= 4x + 2 \\ 5x - 4x &= 2 \quad \therefore x = 2 \end{aligned}$$

Substitute this value of  $x$  in (2)

$$\frac{2y}{4-1} = \frac{y}{2} + \frac{5}{2}$$

$$\frac{2y}{3} = \frac{y}{2} + \frac{5}{2} \quad \text{Multiply by 6.}$$

$$\begin{aligned} 4y &= 3y + 15 \\ 4y - 3y &= 15 \quad \therefore y = 15 \end{aligned}$$

Ans. Distance = 15 miles; rate = 2 miles per hour.

A hare is 50 leaps before a greyhound, and takes 4 leaps to the greyhound's 3, but 2 of the greyhound's leaps are as much as 3 of the hare's; how many leaps must the greyhound take to catch the hare?

Let  $x$  = the number of leaps.

$$2 : x :: 3 : \frac{3x}{2} = \left\{ \begin{array}{l} \text{The number of leaps the hare takes} \\ \text{from the time she starts till over-} \\ \text{taken.} \end{array} \right.$$

$$3 : x :: 4 : \frac{4x}{3} = \left\{ \begin{array}{l} \text{The number of leaps the hare takes} \\ \text{from the time the greyhound} \\ \text{starts.} \end{array} \right.$$

$$\text{Hence } \frac{3x}{2} - 50 = \frac{4x}{3}$$

$$\begin{aligned} 9x - 300 &= 8x \\ x &= 300. \quad \text{Ans.} \end{aligned}$$

A person in play lost a fourth of his money, and then won back 3s., after which he lost a third of what he had left, and then won back 2s.; lastly, he lost a seventh of what he then had, and after this found he had but 12s. remaining: what had he at first?

Let  $x$  = the money he had at first,

then  $\frac{x}{4}$  he lost,

$$x - \frac{x}{4} = \frac{3x}{4} \text{ he had left,}$$

$$\therefore \frac{3x}{4} + 3 \text{ or } \frac{3x + 12}{4} \text{ he had after he won 3s,}$$

$$\frac{x + 4}{4} \text{ he lost the second time,}$$

$$\therefore \frac{3x + 12}{4} - \frac{x + 4}{4} = \frac{2x + 8}{4} = \frac{x + 4}{2} \text{ he had left,}$$

$$\frac{x + 4}{2} + 2 \text{ or } \frac{x + 8}{2} \text{ he had after he won 2s,}$$

$$\text{hence } \frac{x + 8}{14} \text{ he lost the third time,}$$

$$\frac{x + 8}{2} - \frac{x + 8}{14} \text{ or } \frac{7x + 56}{14} - \frac{x + 8}{14} = \frac{6x + 48}{14} \text{ he had left,}$$

$$\text{hence } \frac{6x + 48}{14} = 12 \text{ by the question,}$$

$$\text{or } \frac{x + 8}{14} = 2$$

$$x + 8 = 28$$

$$\therefore x = 28 - 8 = 20s. \text{ Ans.}$$

A company of foot march 1165 of their own paces ahead of a troop of horse; now, if the foot take 5 paces to every 4 of the horse, but 3 paces of a horse be equal in extent to 4 paces of the foot, how many paces will the horse have marched before they overtake the foot?

Let  $x$  be the number of steps the horse must take,

horse steps		horse steps		foot steps		foot steps
then 3	:	$x$	::	4	:	$\frac{4x}{3}$

= number of footsteps the infantry march from the time they start till overtaken.

horse steps		horse steps		foot steps		foot steps
4	:	$x$	::	5	:	$\frac{5x}{4}$

= number of footsteps the infantry march from the time the cavalry start.

$$\therefore \frac{4x}{3} - 1165 = \frac{5x}{4}$$

$$16x - 13980 = 15x$$

$$16x - 15x = 13980$$

$$\therefore x = 13980.$$

A waterman finds by experience that he can, with the advantage of a common tide, row down a river from A to B, which is 18 miles, in an hour and a half; and that to return from B to A against an equal tide, though he rows back along the shore, where the stream is only three-fifths as strong as in the middle, takes him just two hours and a quarter. It is required from hence to find at what rate per hour the tide runs in the middle, where it is strongest.

Let  $x$  = the rate per hour the tide runs in the middle,

then  $\frac{3x}{5}$  = ..... at the side.

and  $\frac{18}{1\frac{1}{2}} = 12$  miles = the rate per hour he can row in the middle, with the advantage of the tide;

$\therefore 12 - x$  = the rate per hour he can row without the tide.

Again,  $\frac{18}{2\frac{1}{4}} = 8$  miles = the rate per hour he can row against the tide at the side;

$\therefore 8 + \frac{3x}{5}$  = the rate he can row without any tide.

$$\text{Hence } 12 - x = 8 + \frac{3x}{5}$$

$$60 - 5x = 40 + 3x$$

$$8x = 20$$

$x = 2\frac{1}{2}$  the rate per hour the tide runs in the middle, where it is strongest.

*Simple Equations*

$$1. \quad \frac{5x}{6} + \frac{2}{3} = x + \frac{1}{3}. \quad x = 2.$$

$$2. \quad \frac{3x-4}{2} = \frac{x}{2} + \frac{x}{4} + 1. \quad x = 4.$$

$$3. \quad 5x - \frac{6x+3}{11} = \frac{7x+15}{2} - 3. \quad x = 5.$$

$$4. \quad \frac{4}{x} + 2 = \frac{24}{x} - \frac{1}{2}. \quad x = 8$$

$$5. \quad \frac{x}{8} - 1 + \frac{x}{12} - \frac{x+5}{4} = -\frac{11}{4}. \quad x = 12.$$

$$6. \quad \frac{x+a}{b} - \frac{x}{a} = 1. \quad x = -a.$$

$$7. \quad \frac{b}{ax} - a^2 = b^2 - \frac{a}{bx}. \quad x = \frac{1}{ab}.$$

$$8. \quad \frac{a(b^2+x^2)}{bx} = ac + \frac{a}{b}x. \quad x = \frac{b}{c}.$$

*Problems.*

1. Ten years ago a boy's age was  $\frac{1}{10}$  of his father's, but now it is  $\frac{1}{4}$  of it; what are both their ages? Ans. 15 and 60.

2. If from three times a certain number we subtract 8, half the remainder will be equal to the number itself diminished by 2; what is the number? Ans. 4.

3. If to the numerator of a certain fraction we add one, its value will be  $\frac{1}{3}$ , but if from its denominator we subtract one, its value will be  $\frac{1}{4}$ ; what is the fraction? Ans.  $\frac{2}{5}$ .

4. £132 is to be divided between A, B, and C, so that B will receive  $\frac{2}{3}$  as much as A, and  $\frac{5}{6}$  as much as A and B together; find the share of each.

Ans. A £40, B £32, C £60.

5. A, having a certain number of sovereigns in his purse, and meeting two of his creditors, B and C, gave to B  $\frac{3}{10}$  of the money, and to C  $\frac{1}{5}$  of the remainder; he then found that he had left exactly  $\frac{1}{10}$  of the original sum; what money had he at first?

Ans £110.

6. A grazier spent  $\frac{1}{12}$  of his money in horses,  $\frac{1}{3}$  in oxen, and  $\frac{3}{10}$  of the remainder in sheep; after which he found he had £98 left: how much had he at first?

Ans. £240.

## CHAPTER III.

## INVOLUTION.

XII. INVOLUTION is the raising of a quantity to any required power by multiplication.

Thus,  $a \times a = a^1 \times a^1 = a^{1+1} = a^2 =$  the 2nd power, or square of  $a$ .

$a \times a \times a = a^{1+1+1} = a^3 =$  the 3rd power, or cube of  $a$ .

$a \times a \times a \times a \dots$  to  $n$  terms  $= a^{1+1+1+1+\&c.}$ , to  $n$  terms  $= a^n$ .

Again,  $a^2 \times a^2 \times a^2 = a^{2+2+2} = a^{2 \times 3} = (a^2)^3 = a^6$ .

$a^n \times a^n \times a^n \dots$  to  $m$  terms  $= a^{n+n+n \dots}$  to  $m$  terms  $= a^{n \times m} = a^{nm}$ .

Hence, to find any power of a quantity, we multiply the index of that quantity by the number indicating the power to which it is to be raised.

$$\sqrt{a} \times \sqrt{a} = a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a,$$

$$\sqrt[3]{a} \times \sqrt[3]{a} = a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}} = a^{\frac{2}{3}} = \sqrt[3]{a^2},$$

$$\sqrt[4]{a} \times \sqrt[4]{a} \times \sqrt[4]{a} = a^{\frac{1}{4}} \times a^{\frac{1}{4}} \times a^{\frac{1}{4}} = a^{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}} = a^{\frac{3}{4}} = \sqrt[4]{a^3}.$$

$$(-a)^2 = -a \times -a = a^2,$$

$$(-a)^3 = -a \times -a \times -a = -a^3,$$

$$(-a)^4 = -a \times -a \times -a \times -a = a^4,$$

&amp;c.

&amp;c.

&amp;c.

Hence, *even* powers of negative quantities are *positive*; and *odd* powers of negative quantities are *negative*.

$$(a+b)^2 = (a+b) \times (a+b) = a^2 + 2ab + b^2$$

$$(a-b)^2 = (a-b) \times (a-b) = a^2 - 2ab + b^2$$

$$\begin{aligned} (a+b)^3 &= (a+b) \times (a+b) \times (a+b) = a^3 + 3a^2b + 3ab^2 + b^3 \\ &= a^3 + b^3 + 3ab(a+b) \end{aligned}$$

$$\begin{aligned} (a-b)^3 &= (a-b) \times (a-b) \times (a-b) = a^3 - 3a^2b + 3ab^2 - b^3 \\ &= a^3 - b^3 - 3ab(a-b). \end{aligned}$$

Hence, by the last two formulæ, the cube of the sum of two quantities is equal to the sum of their cubes + three times their product multiplied by their sum.

The cube of the difference of two quantities is equal to the difference of their cubes — three times their product multiplied by their difference.

$$(a+b+c)^3 = \{(a+b)+c\}^3 = (a+b)^3 + 2(a+b)c + c^3 \\ = a^3 + 2ab + b^3 + 2ac + 2bc + c^3$$

$$(a+b+c+d)^3 = \{(a+b)+(c+d)\}^3 = \\ (a+b)^3 + 2(a+b)(c+d) + (c+d)^3.$$

$$(a+b-c+d)^3 = \{(a+b)-(c-d)\}^3 = \\ (a+b)^3 - 2(a+b)(c-d) + (c-d)^3.$$

### Examples.

1. Show that  $(a-2x)^3 = a^3 - 4ax + 4x^2$ .

2.  $\left(x + \frac{1}{x}\right)^3 = x^3 + 2 + \frac{1}{x^3}$ .

3.  $\left(x + \frac{p}{2}\right)^3 = x^3 + px + \frac{p^3}{4}$ .

4.  $(2x-3y)^3 = 4x^3 - 12xy + 9y^3$ .

5.  $(2x-1)^3 = 4x^3 - 4x + 1$ .

6.  $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ .

7.  $(a-x)^4 = a^4 - 4a^3x + 6a^2x^2 - 4ax^3 + x^4$ .

8.  $(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$ .

9.  $(s-z)^5 = s^5 - 5s^4z + 10s^3z^2 - 10s^2z^3 + 5sz^4 - z^5$ .

10.  $(x^{\frac{1}{2}} + y^{\frac{1}{2}})^2 = x + 2x^{\frac{1}{2}}y^{\frac{1}{2}} + y = x + 2\sqrt{xy} + y$ .

11.  $(x - \sqrt{y})^2 = x^2 - 2x\sqrt{y} + y$ .

12.  $(x^2y + \sqrt{y})^2 = x^4y^3 + 2x^2y^{\frac{3}{2}} + y$ .

13.  $(x^{\frac{1}{2}} - y^{\frac{1}{2}})^3 = x - 3x^{\frac{1}{2}}y^{\frac{1}{2}} + 3x^{\frac{1}{2}}y^{\frac{3}{2}} - y$ .

14.  $(x+1)^4 = x^4 + 4x^3 + 6x^2 + 4x + 1$ .

15.  $\{(a+b)^{\frac{1}{2}} - (a-b)^{\frac{1}{2}}\} = \\ 2b - 3(a+b)^{\frac{1}{2}}(a-b)^{\frac{1}{2}}\{(a+b)^{\frac{1}{2}} - (a-b)^{\frac{1}{2}}\}.$

### EVOLUTION.

XIII. EVOLUTION is the extraction of roots.

The square of  $a$  being  $a^2$ , the square root of  $a^2$  is  $a$ .

The cube of  $a$  being  $a^3$ , the cube root of  $a^3$  is  $a$ .

The square root of  $a$  is  $\sqrt{a}$ , or  $a^{\frac{1}{2}}$ ;  $\therefore \sqrt{a} \times \sqrt{a} = a$ , and  $a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2} + \frac{1}{2}} = a^1 = a$ .

The cube root of  $a$  is  $\sqrt[3]{a}$ , or  $a^{\frac{1}{3}}$ .

The  $n$ th root of  $a$  is  $\sqrt[n]{a}$ , or  $a^{\frac{1}{n}}$ .

The cube root of  $a^3$  is  $\sqrt[3]{a^3}$ , or  $a^{\frac{3}{3}}$ .

The  $m$ th root of  $a^n$  is  $\sqrt[m]{a^n}$ , or  $a^{\frac{n}{m}}$ .

Hence, to find any root of a quantity, we divide the index by the number indicating the root required;

$$\therefore +a \times +a = +a^2, \text{ and } -a \times -a = +a^2$$

$$\therefore \sqrt{a^2} = +a, \text{ or } -a.$$

Hence the square root of a quantity is either positive, or negative.

$$\therefore +a \times +a \times +a = +a^3, \text{ but } -a \times -a \times -a = -a^3;$$

$$\therefore \sqrt[3]{a^3} = +a, \text{ but } \sqrt[3]{-a^3} = -a.$$

Hence the cube root of a positive quantity is positive, but the cube root of a negative quantity is negative.

Since there is no quantity, which, being multiplied by itself, will produce  $-a$ ,  $\therefore \sqrt{-a}$  is an *impossible* quantity.

$a^2 + 2ab + b^2$  being the square of  $a + b$ , we may, by means of this formula, investigate a rule for extracting the square root of a binomial; for  $a^2 + 2ab + b^2 = a^2 + (2a + b)b$ , and  $\therefore$  the first term of the root is  $a$ , the square root of  $a^2$ , we have only to find a divisor which will give the quotient  $+b$ , and this divisor is evidently  $2a + b$ ; that is, twice the first term of the root together with the second term of the root.

Hence, to find the square root of  $a^2 + 2ab + b^2$ , we first take  $a$ , the square root of  $a^2$ , and place it in the root, and subtract its square from  $a^2 + 2ab + b^2$ ; we then double the root  $a$  for a divisor, and divide the remaining portion of the quantity, namely  $(2a + b)b$ , by  $2a$ , and place the quotient  $b$  in the root, and also annex it to the divisor; we then multiply  $2a + b$  by the second term  $b$ , and subtract the result.

The process may be thus exhibited,

$$\begin{array}{r}
 a^3 + 2ab + b^3 \quad (a + b \\
 \underline{a^3} \\
 2a + b) \quad \begin{array}{r} 2ab + b^3 \\ 2ab + b^3 \end{array} \\
 \hline
 \end{array}$$

Let us apply this rule to the extraction of the square root of the number 1225.

$$\begin{array}{r}
 1225 \quad (30 + 5, \text{ or } 35 \\
 \underline{900^*} \\
 60 + 5, \text{ or } 65) \quad \begin{array}{r} 325 \\ 325 \end{array} \\
 \hline
 \end{array}$$

Let it be required to extract the square root of  $a^3 + 2ab + 2ac + 2bc + b^3 + c^3$ .

$$\begin{array}{r}
 a^3 + 2ab + 2ac + 2bc + b^3 + c^3 \quad (a + b + c \\
 \underline{a^3} \\
 2a + b) \quad \begin{array}{r} 2ab \\ 2ab \end{array} \quad \begin{array}{r} + b^3 \\ + b^3 \end{array} \\
 \hline
 2a + 2b + c) \quad \begin{array}{r} 2ac + 2bc \\ 2ac + 2bc \end{array} \quad \begin{array}{r} + c^3 \\ + c^3 \end{array} \\
 \hline
 \end{array}$$

In this example, having obtained  $a + b$ , this portion of the root is doubled for a new divisor, in order to obtain  $+ c$ , the remaining portion of the root.

Since the cube root of  $a^3 + 3a^2b + 3ab^2 + b^3$  is  $a + b$ , we may readily derive a rule for the extraction of the cube root. The cube root of the first term  $a^3$ , is  $a$ , the first term of the root; we subtract its cube from the whole quantity; then, for a divisor we take  $3a^2$ , and dividing the first term of the remainder by it, we obtain  $b$ , the second term of the root; then annexing  $3ab + b^2$  to the divisor, and multiplying it by  $b$ , we obtain  $3a^2b + 3ab^2 + b^3$ , which is equal to the remainder.

\* In the ordinary arithmetical process these two ciphers would be omitted.



The process is as follows,

$$\begin{array}{r}
 a^3 + 3a^2b + 3ab^2 + b^3 \quad (a + b \\
 \underline{a^3} \\
 3a^2) \quad 3a^2b + 3ab^2 + b^3 \\
 \underline{3a^2b + 3ab^2 + b^3}
 \end{array}$$

We will apply this rule to the extraction of the cube root of 91125.

$$\begin{array}{r}
 91125 \quad \begin{smallmatrix} a & b \\ (40 + 5, \text{ or } 45 \end{smallmatrix} \\
 \underline{a^3 = 64000} \\
 3a^2 = 3 \times 40^2 = 4800) \quad 27125 \\
 \underline{3a^2b = 3 \times 40^2 \times 5 = 24000} \\
 3ab^2 = 3 \times 40 \times 5^2 = 3000 \\
 \underline{b^3 = 5^3 = 125} \\
 27125
 \end{array}$$

The ordinary arithmetical process would be

$$\begin{array}{r}
 91125 \quad (45 \\
 \underline{64} \\
 4^2 \times 3 = 48) \quad 27125 \\
 \underline{5^3 = 125} \\
 5^2 \times 4 \times 3 = 300 \\
 \underline{5 \times 48 = 240} \\
 27125
 \end{array}$$

### Examples.

1. Find the square root of  $4a^2 + 4ax + x^2$ .

$$\begin{array}{r}
 4a^2 + 4ax + x^2 \quad (2a + x = \text{the root.} \\
 \underline{4a^2}
 \end{array}$$

$$\begin{array}{r}
 4a + x) \quad 4ax + x^2 \\
 \underline{4ax + x^2}
 \end{array}$$

$$2. \sqrt{(29x^2y^2 + 30xy^3 + 12x^3y + 25y^4 + 4x^4)}.$$

$$\begin{array}{r} 9x^2y^2 + 20x^2y^2 + 30xy^3 + 12x^3y + 25y^4 + 4x^4 \\ 9x^2y^2 \qquad \qquad \qquad (3xy + 2x^2 + 5y^2). \end{array}$$

$$\begin{array}{r} 6xy + 2x^2 \qquad \qquad 12x^3y \qquad \qquad \qquad + 4x^4 \\ \qquad \qquad \qquad 12x^3y \qquad \qquad \qquad + 4x^4 \\ \hline \end{array}$$

$$\begin{array}{r} 6xy + 4x^2 + 5y^2 \qquad \qquad 30xy^3 + 20x^2y^2 + 25y^4 \\ \qquad \qquad \qquad 30xy^3 + 20x^2y^2 + 25y^4 \\ \hline \end{array}$$

In this example  $29x^2y^2 = 9x^2y^2 + 20x^2y^2$ , and the other terms are brought down in the order which is found to be most convenient.

$$3. \sqrt{a^2 - x^2}.$$

$$a^2 - x^2 \left( a - \frac{x^2}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5} - \&c. = \text{the root.} \right.$$

$$\frac{a^2}{\quad}$$

$$2a - \frac{x^2}{2a} \quad \left. \right) - x^2$$

$$-x^2 + \frac{x^4}{4a^2}$$

$$2a - \frac{x^2}{a} - \frac{x^4}{8a^3} \quad \left. \right) - \frac{x^4}{4a^2}$$

$$-\frac{x^4}{4a^2} + \frac{x^6}{8a^4} + \frac{x^8}{64a^6}$$

$$2a - \frac{x^2}{a} - \frac{x^4}{4a^3} \quad \left. \right) - \frac{x^6}{8a^4} - \frac{x^8}{64a^6}$$

Here the process does not terminate, and the root is merely an approximate value of  $\sqrt{a^2 - x^2}$ .

$$4. \text{ Extract the cube root of } 8a^3 - 84a^2x + 294ax^2 - 343x^3.$$

$$\frac{8a^3 - 84a^2x + 294ax^2 - 343x^3}{8a^3} (2a - 7x)$$

$$(2a)^2 \times 3 = 12a^2) \quad \frac{-84a^2x + 294ax^2 - 343x^3}{-84a^2x}$$

$$\left( \frac{-7x}{-7x} \right)^2 \times 3 =$$

$$\left( \frac{-7x}{-7x} \right)^2 \times 2a \times 3 =$$

$$\left( \frac{-7x}{-7x} \right)^3 =$$

$$+ 294ax^2$$

$$- 343x^3$$

$$\frac{-84a^2x + 294ax^2 - 343x^3}{-84a^2x + 294ax^2 - 343x^3}$$

$$b. \sqrt[3]{(a^3 + 3a^2b + 3a^2c + 3ab^2 + 6abc + 3ac^2 + b^3 + 3b^2c + 3bc^2 + c^3)}.$$

$$a^3 + 3a^2b + 3a^2c + 3ab^2 + 6abc + 3ac^2 + b^3 + 3b^2c + 3bc^2 + c^3 (a + b + c).$$

$$\frac{3a^2}{a^2} \sqrt[3]{\frac{3a^2b}{3a^2b}}$$

$$b \times a^2 \times 3 = 3a^2b$$

$$b^3 \times a \times 3 =$$

$$b^3 =$$

$$3ab^3$$

$$+ b^3$$

$$(a+b)^2 \times 3 = 3a^2 + 6ab + 3b^2) \quad \frac{3a^2c}{3a^2c} \quad + 6abc + 3ac^2 \quad + 3b^2c + 3bc^2 + c^3$$

$$c \times (a+b)^2 \times 3 =$$

$$c^3 \times (a+b) \times 3 =$$

$$c^3 =$$

$$+ 6abc$$

$$+ 3ac^2$$

$$+ 3bc^2$$

$$+ c^3$$

6. The square root of  $4x^2 - 4ax + a^2 = 2x - a$ .

7.  $\sqrt{1 - 8a + 16a^2} = 1 - 4a$ ; and  $\sqrt{25a^2 + 40ax + 16x^2} = 5a + 4x$ .

$$8. \sqrt{9x^2 - 3x + \frac{1}{4}} = 3x - \frac{1}{2}.$$

$$9. \sqrt{49a^4 + 42a^2b + 9b^2} = 7a^2 + 3b.$$

$$10. \sqrt{(x^2 - 2ax + a^2 + 2x - 2a + 1)} = x - a + 1.$$

$$11. \sqrt{\left(m^2 + 2m - 1 - \frac{2}{m} + \frac{1}{m^2}\right)} = m + 1 - \frac{1}{m}.$$

$$12. \sqrt{\left(\frac{a^2}{x^2} - 2 + \frac{x^2}{a^2} + \frac{2a^2}{x} - 2x + a^2\right)} = \frac{a}{x} - \frac{x}{a} + a.$$

$$13. \sqrt[3]{(a^3 - 3a^2x + 3ax^2 - x^3)} = a - x.$$

$$14. \sqrt[3]{(1 - 12a + 48a^2 - 64a^3)} = 1 - 4a.$$

$$15. \sqrt[3]{(8a^3 + 36a^2x + 54ax^2 + 27x^3)} = 2x + 3a.$$

$$16. \sqrt[3]{(x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1)} = x^2 - 2x + 1.$$

$$17. \sqrt[3]{(y^6 - 6y^5 + 6y^4 + 16y^3 - 12y^2 - 24y - 8)} = y^2 - 2y - 2.$$

## CHAPTER IV.

### SURDS.

XIV. SURDS, or Irrational quantities, are such that the roots indicated cannot be exactly determined; thus,

$\sqrt{2}$ ,  $\sqrt{5}$ ,  $\sqrt[3]{4}$ ,  $\sqrt[3]{3^2}$ ,  $\sqrt{1+x^2}$ ,  $\sqrt{x^2+ax}$ , and  $\sqrt[3]{2x^2}$  are surds.

These quantities might be expressed by using fractional indices, thus,  $2^{\frac{1}{2}}$ ,  $5^{\frac{1}{2}}$ ,  $4^{\frac{1}{3}}$ ,  $3^{\frac{2}{3}}$ ,  $(1+x^2)^{\frac{1}{2}}$ ,  $(x^2+ax)^{\frac{1}{2}}$ , and  $(2x^2)^{\frac{1}{3}}$ .

Any rational quantity may be made to assume the form of a surd; thus,  $x = \sqrt{x^2} = x^{\frac{1}{2}}$ ,  $x - a = \sqrt{(x-a)^2} = (x-a)^{\frac{2}{2}}$ ,  $a\sqrt{x} = a^{\frac{1}{2}}x^{\frac{1}{2}} = (a^2x)^{\frac{1}{2}} = \sqrt{a^2x}$ .

*Examples.*

$$1. \text{ Prove that } \sqrt{243} + \sqrt{27} + \sqrt{4} = 16\sqrt{3}.$$

$$\sqrt{243} = \sqrt{81 \times 3} = 9\sqrt{3}$$

$$\sqrt{27} = \sqrt{9 \times 3} = 3\sqrt{3}$$

$$\sqrt{4} = \sqrt{16 \times 3} = 4\sqrt{3}$$

$$\therefore \sqrt{248} + \sqrt{27} + \sqrt{48} = 9\sqrt{8} + 3\sqrt{3} + 4\sqrt{3} \\ = (9+3+4)\sqrt{3} = 16\sqrt{3}.$$

2. Show that  $2\sqrt{8a^3} - 7a\sqrt{18a} + 5\sqrt{72a^3} - \sqrt{50ab^2}$   
 $= (13a - 5b)\sqrt{2a}.$

$$2\sqrt{8a^3} = 2\sqrt{4a^2 \times 2a} = 2 \times 2a\sqrt{2a} = 4a\sqrt{2a} \\ - 7a\sqrt{18a} = -7a\sqrt{9 \times 2a} = -7a \times 3\sqrt{2a} = -21a\sqrt{2a} \\ 5\sqrt{72a^3} = 5\sqrt{36a^2 \times 2a} = 5 \times 6a\sqrt{2a} = 30a\sqrt{2a} \\ - \sqrt{50ab^2} = -\sqrt{25b^2 \times 2a} = -5b\sqrt{2a};$$

and  $4a\sqrt{2a} - 21a\sqrt{2a} + 30a\sqrt{2a} - 5b\sqrt{2a} =$   
 $(4a - 21a + 30a - 5b)\sqrt{2a} = (13a - 5b)\sqrt{2a}.$

3. Multiply

$$\left(4\sqrt{\frac{7}{3}} + 5\sqrt{\frac{1}{2}}\right) \text{ by } \left(\sqrt{\frac{7}{3}} + 2\sqrt{\frac{1}{2}}\right).$$

This operation can be performed in the same way as the multiplication of ordinary algebraic quantities; thus,

$$\begin{array}{r} 4\sqrt{\frac{7}{3}} + 5\sqrt{\frac{1}{2}} \\ \sqrt{\frac{7}{3}} + 2\sqrt{\frac{1}{2}} \\ \hline 4 \times \frac{7}{3} + 5\sqrt{\frac{7}{6}} \\ \quad 8\sqrt{\frac{7}{6}} + 10 \times \frac{1}{2} \\ \hline \frac{28}{3} + 13\sqrt{\frac{7}{6}} + 5 \\ = \frac{28}{3} + \frac{15}{3} + 13\sqrt{\frac{7}{6}} = \frac{43}{3} + 13\sqrt{\frac{7}{6}} \end{array}$$

4. Prove that  $12 \sqrt[3]{\frac{1}{4}} + 3 \sqrt[3]{\frac{1}{32}} = \frac{27}{4} \sqrt[3]{2}$

$$12 \sqrt[3]{\frac{1}{4}} = 12 \sqrt[3]{\frac{2}{8}} = \frac{12}{2} \sqrt[3]{2}$$

$$3 \sqrt[3]{\frac{1}{32}} = 3 \sqrt[3]{\frac{2}{64}} = \frac{3}{4} \sqrt[3]{2};$$

$$\text{and } \frac{12}{2} \sqrt[3]{2} + \frac{3}{4} \sqrt[3]{2} = \frac{24}{4} \sqrt[3]{2} + \frac{3}{4} \sqrt[3]{2} = \frac{27}{4} \sqrt[3]{2}.$$

5. Reduce  $(2\sqrt{8} + 3\sqrt{5} - 7\sqrt{2})$  and  $(\sqrt{72} - 5\sqrt{20} - 2\sqrt{2})$  to their simplest form, and find their product.

$$2\sqrt{8} = 2\sqrt{4 \times 2} = 2 \times 2\sqrt{2} = 4\sqrt{2}$$

$$\therefore 2\sqrt{8} + 3\sqrt{5} - 7\sqrt{2} = 4\sqrt{2} + 3\sqrt{5} - 7\sqrt{2} \\ = 3\sqrt{5} - 3\sqrt{2}$$

$$\sqrt{72} = \sqrt{36 \times 2} = 6\sqrt{2};$$

$$\text{and } -5\sqrt{20} = -5\sqrt{4 \times 5} = -10\sqrt{5}$$

$$\therefore \sqrt{72} - 5\sqrt{20} - 2\sqrt{2} = 6\sqrt{2} - 10\sqrt{5} \\ - 2\sqrt{2} = 4\sqrt{2} - 10\sqrt{5};$$

$$3\sqrt{5} - 3\sqrt{2}$$

$$4\sqrt{2} - 10\sqrt{5}$$

$$12\sqrt{10} - 12 \times 2$$

$$- 30 \times 5 + 30\sqrt{10}$$

$$42\sqrt{10} - 24 - 150 = 42\sqrt{10} - 174.$$

6. Express as a single fraction

$$\frac{a + b\sqrt{-1}}{a - b\sqrt{-1}} + \frac{a - b\sqrt{-1}}{a + b\sqrt{-1}}.$$

Reducing these fractions to a common denominator,

$$\frac{a + b\sqrt{-1}}{a - b\sqrt{-1}} \times \frac{a + b\sqrt{-1}}{a + b\sqrt{-1}} = \frac{a^2 + 2ab(-1) + b^2(-1)}{a^2 - b^2(-1)}$$

$$= \frac{a^2 - 2ab - b^2}{a^2 + b^2};$$

$$\frac{a - b\sqrt{-1}}{a + b\sqrt{-1}} \times \frac{a - b\sqrt{-1}}{a - b\sqrt{-1}} = \frac{a^2 - 2ab(-1) + b^2(-1)}{a^2 - b^2(-1)}$$

$$= \frac{a^2 + 2ab - b^2}{a^2 + b^2}.$$

Hence by addition,

$$\frac{a + b\sqrt{-1}}{a - b\sqrt{-1}} + \frac{a - b\sqrt{-1}}{a + b\sqrt{-1}} = \frac{2a^2 - 2b^2}{a^2 + b^2} = \frac{2(a^2 - b^2)}{a^2 + b^2}.$$

This example exhibits a method of making a binomial surd rational; for the denominators of the given fractions are each multiplied by a quantity which has made them rational. In general since  $(x + y)(x - y) = x^2 - y^2$ , any surd of the form  $a \pm \sqrt{b}$  or  $\sqrt{a} \pm \sqrt{b}$  may be made rational by multiplying by a similar surd of the form  $a \mp \sqrt{b}$  or  $\sqrt{a} \mp \sqrt{b}$  using the upper sign when the *sum* is given, and the lower when the *difference* is given.

7. Show that

$$\frac{1}{\sqrt{2} + \sqrt{3} - \sqrt{5}} = \frac{\sqrt{30} + 3\sqrt{2} + 2\sqrt{3}}{12}.$$

Multiplying both terms of the given fraction by

$\sqrt{2} + \sqrt{3} + \sqrt{5}$ , we have

$$\frac{1}{\sqrt{2} + \sqrt{3} - \sqrt{5}} \times \frac{\sqrt{2} + \sqrt{3} + \sqrt{5}}{\sqrt{2} + \sqrt{3} + \sqrt{5}}$$

$$= \frac{\sqrt{2} + \sqrt{3} + \sqrt{5}}{2\sqrt{6}}$$

$\frac{28}{3} +$  multiplying both terms by  $\sqrt{6}$ , we have

$$= \frac{28}{3} + \frac{15}{3} + \frac{18 + \sqrt{30}}{6} = \frac{2\sqrt{3} + 3\sqrt{2} + \sqrt{30}}{12}.$$

8. Multiply  $a^{\frac{1}{2}} + a^{\frac{1}{2}}b^{\frac{1}{2}} + a^{\frac{1}{2}}b + b^{\frac{1}{2}}$  by  $a^{\frac{1}{2}} - b^{\frac{1}{2}}$ .

$$\begin{array}{r} a^{\frac{1}{2}} + a^{\frac{1}{2}}b^{\frac{1}{2}} + a^{\frac{1}{2}}b + b^{\frac{1}{2}} \\ a^{\frac{1}{2}} - b^{\frac{1}{2}} \\ \hline a + a^{\frac{1}{2}}b^{\frac{1}{2}} + a^{\frac{1}{2}}b + a^{\frac{1}{2}}b^{\frac{1}{2}} \\ - a^{\frac{1}{2}}b^{\frac{1}{2}} - a^{\frac{1}{2}}b - a^{\frac{1}{2}}b^{\frac{1}{2}} - b^2 \\ \hline a \quad * \quad * \quad * \quad - b^2. \end{array}$$

9. Show that

$$\begin{aligned} \sqrt{\left\{ 1 + \frac{11}{4}a - \sqrt{3}a(1+a) + a^2 \right\}} &= 1 - \frac{\sqrt{3}}{2}\sqrt{a} + a. \\ 1 + \frac{11}{4}a - \sqrt{3}a^{\frac{1}{2}} - \sqrt{3}a^{\frac{3}{2}} + a^2 &\left( 1 - \frac{\sqrt{3}}{2}a^{\frac{1}{2}} + a. \right. \\ &\left. 2 - \frac{\sqrt{3}}{2}a^{\frac{1}{2}} \right) \frac{1}{2} - \sqrt{3} \cdot a^{\frac{1}{2}} + \frac{11}{4}a \\ &\quad - \sqrt{3} \cdot a^{\frac{1}{2}} + \frac{3}{4}a \\ &\hline 2 - \sqrt{3} \cdot a^{\frac{1}{2}} + a \quad \quad \quad \begin{array}{l} 2a - \sqrt{3} \cdot a^{\frac{3}{2}} + a^2 \\ 2a - \sqrt{3} \cdot a^{\frac{3}{2}} + a^2 \\ * \quad * \quad * \end{array} \end{aligned}$$

10. Show that  $\sqrt{12} + \sqrt{27} - \sqrt{3} + \sqrt{48} = 8\sqrt{3}$

11. Show that  $\sqrt[3]{40} - 3\sqrt[3]{320} + 4\sqrt[3]{195} = 2\sqrt[3]{5}$

12. Show that  $\sqrt[3]{16} - 6\sqrt[3]{\frac{1}{4}} + 2\sqrt[3]{54} - 4\sqrt[3]{\frac{1}{32}} = 4\sqrt[3]{2}$ .

13.  $2\sqrt{\frac{a}{2}} - b\sqrt{\frac{a}{b}} = \sqrt{2a} - \sqrt{ab}.$

14.  $5\sqrt[3]{a} - (4a)^{\frac{1}{3}} - 3a^{\frac{1}{3}} + \sqrt[3]{16a} = 2a^{\frac{1}{3}}(a+1).$

15.  $2\sqrt[3]{2x} + 6\sqrt[6]{4x^2} + \sqrt[9]{8x^3} = 9\sqrt[3]{2x}.$



$$16. \sqrt{18a^5b^3} + \sqrt{50a^3b^3} = (3a^2b + 5ab)\sqrt{2ab}.$$

$$17. \sqrt[3]{4} \times 7\sqrt[3]{6} \times \sqrt[6]{\frac{5}{8}} = 7\sqrt[6]{15}.$$

$$18. (\sqrt{5} + 1)(1 - \sqrt{5}) = -4; \text{ and} \\ (\sqrt{3} + \sqrt{2})(\sqrt{3} - \sqrt{2}) = 1.$$

$$19. \frac{\sqrt{a} + \sqrt{b}}{\sqrt{a} - \sqrt{b}} = \frac{a + 2a^{\frac{1}{2}}b^{\frac{1}{2}} + b}{a - b}; \text{ and} \\ \sqrt[6]{\frac{1}{x^3y^2}} \times \sqrt[5]{x^3} \cdot \sqrt[3]{y} = \frac{1}{x^{\frac{1}{15}}}$$

$$20. \frac{\sqrt{3} - \sqrt{2}}{\sqrt{2} + 1} = \sqrt{2} - \sqrt{3} + \sqrt{6} - 2; \text{ and} \\ (\sqrt{3} + \sqrt{2})^3 = 9\sqrt{3} + 11\sqrt{2}.$$

$$21. \frac{1}{x - \sqrt{x^2 - 1}} + \frac{1}{x + \sqrt{x^2 + 1}} = 2x; \text{ and} \\ \frac{\sqrt{x+y} + \sqrt{x-y}}{\sqrt{x+y} - \sqrt{x-y}} = \frac{x}{y} + \frac{\sqrt{x^2 - y^2}}{y}.$$

$$22. \text{ Prove that } \frac{\sqrt{x^2 - a^2}}{2} + \frac{x}{2} \text{ is the square root of}$$

$$\frac{x}{2}\sqrt{x^2 - a^2} + \frac{a^2}{4}; \text{ and that}$$

$$\left\{ \sqrt{\frac{x + \sqrt{x^2 - a^2}}{2}} + \sqrt{\frac{x - \sqrt{x^2 - a^2}}{2}} \right\}^2 = x + \frac{a}{2}.$$

## CHAPTER V.

### SIMPLE EQUATIONS.

$$1. \sqrt{x + 16} = 2 + \sqrt{x}; \text{ find } x.$$

Squaring each member of the equation, we have

$$\begin{aligned}
 x + 16 &= 4 + 4\sqrt{x} + x, \\
 16 - 4 &= 4\sqrt{x}, \quad 12 = 4\sqrt{x}, \\
 3 &= \sqrt{x}, \quad \therefore 9 = x.
 \end{aligned}$$

$$2. \quad \frac{x-a}{\sqrt{a}+\sqrt{x}} = \frac{\sqrt{x}-\sqrt{a}}{3} + 2\sqrt{a}; \text{ find } x.$$

First, reducing the left-hand member of the equation,

$$\frac{x-a}{\sqrt{a}+\sqrt{x}} = \frac{(\sqrt{x}+\sqrt{a})(\sqrt{x}-\sqrt{a})}{(\sqrt{x}+\sqrt{a})} = (\sqrt{x}-\sqrt{a});$$

$$\therefore (\sqrt{x}-\sqrt{a}) = \frac{\sqrt{x}-\sqrt{a}}{3} + 2\sqrt{a},$$

$$3(\sqrt{x}-\sqrt{a}) = (\sqrt{x}-\sqrt{a}) + 6\sqrt{a},$$

$$2(\sqrt{x}-\sqrt{a}) = 6\sqrt{a}, \quad \sqrt{x}-\sqrt{a} = 3\sqrt{a},$$

$$\sqrt{x} = 4\sqrt{a}, \quad \therefore x = 16a.$$

$$3. \quad \frac{x}{\sqrt{a^2+x^2}} = \frac{c-x}{\sqrt{b^2+(c-x)^2}}; \text{ find } x.$$

Squaring each member, we have

$$\frac{x^2}{a^2+x^2} = \frac{(c-x)^2}{b^2+(c-x)^2};$$

Multiply by the product of the denominators, which will thus both be cancelled.

$$b^2x^2 + (c-x)^2x^2 = a^2(c-x)^2 + (c-x)^2x^2,$$

$$b^2x^2 = a^2(c-x)^2, \quad bx = a(c-x) = ac - ax,$$

$$ax + bx = ac, \quad (a+b)x = ac,$$

$$\therefore x = \frac{ac}{a+b}.$$

4. Given  $1 - \sqrt{1-x} = n(1 + \sqrt{1-x})$ ; find  $x$ .

$$1 - \sqrt{1-x} = n + n\sqrt{1-x},$$

$$1 - n = n\sqrt{1-x} + \sqrt{1-x}$$

$$= (n+1)\sqrt{1-x},$$

$$1 - 2n + n^2 = (n^2 + 2n + 1)(1-x)$$

$$= n^2 + 2n + 1 - (n^2 + 2n + 1)x,$$

$$(n+1)^2 x = 4n, \quad \therefore x = \frac{4n}{(n+1)^2}.$$

5.  $x - 4 = \frac{x^2}{(1 + \sqrt{1+x})^2}$ ; find  $x$ .

$$\sqrt{x-4} = \frac{x}{\sqrt{1+x}+1}$$

$$= \frac{x(\sqrt{1+x}-1)}{(\sqrt{1+x}+1)(\sqrt{1+x}-1)} = \frac{x(\sqrt{1+x}-1)}{1+x-1}, \text{ Art. XIV.}$$

$$= \frac{x(\sqrt{1+x}-1)}{x} = \sqrt{1+x}-1,$$

$$\therefore x-4 = 1+x-2\sqrt{1+x}+1,$$

$$2\sqrt{1+x} = 6, \quad \sqrt{1+x} = 3,$$

$$1+x = 9, \quad \therefore x = 8.$$

6.  $8\sqrt{3x} + \frac{243 + 324\sqrt{3x}}{16x-3} = 16x + 3,$

$$16x - 8\sqrt{3}\sqrt{x} + 3 = \frac{81(3 + 4\sqrt{3}\sqrt{x})}{16x-3}$$

$$= \frac{81\sqrt{3}(\sqrt{3} + 4\sqrt{x})}{(4\sqrt{x} + \sqrt{3})(4\sqrt{x} - \sqrt{3})}$$

$$(4\sqrt{x} - \sqrt{3})^2 = \frac{81\sqrt{3}}{(4\sqrt{x} - \sqrt{3})},$$

$$(4\sqrt{x} - \sqrt{3})^3 = 3^4 \cdot 3^{\frac{1}{2}} = 3^{\frac{9}{2}},$$

$$4\sqrt{x} - \sqrt{3} = 3^{\frac{3}{2}}, \quad 4\sqrt{x} = 3^{\frac{3}{2}} + \sqrt{3},$$

$$16x = 3^3 + 2 \cdot 3^3 + 3 = 27 + 18 + 3 = 48, \quad \therefore x = 3.$$

$$7. \quad \sqrt[3]{a+x} + \sqrt[3]{a-x} = \sqrt[3]{b}; \text{ find } x.$$

$$(a+x)^{\frac{1}{3}} + (a-x)^{\frac{1}{3}} = b^{\frac{1}{3}}; \text{ and, cubing each member,}$$

$$a+x+a-x+3\{(a+x)^{\frac{1}{3}} + (a-x)^{\frac{1}{3}}\}(a+x)^{\frac{1}{3}}(a-x)^{\frac{1}{3}} = b,$$

$$3\{(a+x)^{\frac{1}{3}} + (a-x)^{\frac{1}{3}}\}(a+x)^{\frac{1}{3}}(a-x)^{\frac{1}{3}} = b - 2a,$$

$$3b^{\frac{1}{3}}(a+x)^{\frac{1}{3}}(a-x)^{\frac{1}{3}} = b - 2a,$$

$$\therefore (a+x)^{\frac{1}{3}} + (a-x)^{\frac{1}{3}} = b^{\frac{1}{3}},$$

$$\sqrt[3]{(a+x)(a-x)} = \frac{b - 2a}{3b^{\frac{1}{3}}},$$

$$(a+x)(a-x) = \frac{(b - 2a)^3}{27b}, \quad a^2 - x^2 = \frac{(b - 2a)^3}{27b},$$

$$x^2 = a^2 - \frac{(b - 2a)^3}{27b}, \quad \therefore x = \sqrt{a^2 - \frac{(b - 2a)^3}{27b}}.$$

$$8. \quad x\sqrt{x^4-1} + \sqrt[4]{x^4-1} = x^3.$$

$$\sqrt[4]{x^4-1} = x^3 - x\sqrt{x^4-1},$$

$$\sqrt{x^4-1} = x^6 - 2x^4\sqrt{x^4-1} + x^6 - x^3,$$

$$\begin{aligned} x^3 + \sqrt{x^4-1} &= 2x^6 - 2x^4\sqrt{x^4-1} \\ &= 2x^4(x^2 - \sqrt{x^4-1}). \end{aligned}$$

Multiply by  $(x^2 + \sqrt{x^4-1})$ ;

$$(x^2 + \sqrt{x^4-1})^2 = 2x^4(x^2 - \sqrt{x^4-1}) = 2x^4,$$

$$x^2 + \sqrt{x^4-1} = x^2\sqrt{2}, \quad \sqrt{x^4-1} = x^2(\sqrt{2}-1),$$

$$x^4 - 1 = x^4(3 - 2\sqrt{2}) = 3x^4 - 2\sqrt{2} \cdot x^4,$$

$$(2\sqrt{2} - 2)x^4 = 1, \quad x^4 = \frac{1}{2(\sqrt{2}-1)}.$$

Multiply both terms of the fraction by  $\sqrt{2} + 1$ .

$$x^4 = \frac{\sqrt{2} + 1}{2}; \quad \therefore x = \sqrt[4]{\frac{\sqrt{2} + 1}{2}}.$$

9.  $\frac{a+x}{\sqrt{a} + \sqrt{a+x}} + \frac{a-x}{\sqrt{a} + \sqrt{a-x}} = \sqrt{a}$ ; find  $x$ .

$$\frac{(a+x)(\sqrt{a} - \sqrt{a+x})}{a - a + x} + \frac{(a-x)(\sqrt{a} - \sqrt{a-x})}{a - a - x} = \sqrt{a},$$

$$\frac{(a+x)(\sqrt{a} - \sqrt{a+x})}{-x} + \frac{(a-x)(\sqrt{a} - \sqrt{a-x})}{x} = \sqrt{a},$$

$$(a-x)(\sqrt{a} - \sqrt{a-x}) - (a+x)(\sqrt{a} - \sqrt{a+x}) = \sqrt{a} \cdot x,$$

$$\sqrt{a} - \sqrt{a} \cdot x - (a-x)^{\frac{1}{2}} - a\sqrt{a} - \sqrt{a} \cdot x + (a+x)^{\frac{1}{2}} = \sqrt{a} \cdot x,$$

$$(a+x)^{\frac{1}{2}} - (a-x)^{\frac{1}{2}} = 3\sqrt{a} \cdot x,$$

$$(a+x)^3 + (a-x)^3 - 2(a^2 - x^2)^{\frac{3}{2}} = 9ax^2,$$

$$2a^3 + 6ax^2 - 9ax^2 = 2(a^3 - x^3)^{\frac{3}{2}},$$

$$2a^3 - 3ax^2 = 2(a^3 - x^3)^{\frac{3}{2}},$$

$$4a^6 - 12a^4x^2 + 9a^2x^4 = 4(a^3 - x^3)^3,$$

$$= 4a^6 - 12a^4x^2 + 12a^2x^4 - 4x^6,$$

$$4x^6 = 3a^2x^4, \quad 4x^2 = 3a^2,$$

$$2x = a\sqrt{3}, \quad \therefore x = \frac{a}{2}\sqrt{3}.$$

XV. In solving equations involving *two* unknown quantities, as  $x$  and  $y$ , we may adopt any one of the three following methods, namely;

1. Eliminate one of the unknown quantities, and from the resulting equation, determine the value of the other.

2. Find a value of one of the unknown quantities, as  $x$ , from each equation, and then equate these two values.

3. Find a value of one of the unknown quantities, as  $x$ , and substitute it in the other equation.

1. Given  $5x + 3y = 74$  } Find the values of  $x$  and  $y$  by  
 $3x + 2y = 46$  } elimination.

Multiplying the first equation by 2, the coefficient of  $y$  in the second; and multiplying the second by 3, the coefficient of  $y$  in the first, we have

$$\begin{aligned} 10x + 6y &= 148, \\ 9x + 6y &= 138. \end{aligned}$$

Now, since  $y$  has the same coefficient and the same sign in each equation, if we *subtract*\* the lower equation from the upper,  $y$  will disappear, and we shall have

$$x = 10; \quad \therefore 3x = 30.$$

And, writing 30 instead of  $3x$  in the second given equation,  
 $30 + 2y = 46; \quad 2y = 16, \quad \therefore y = 8.$

When three or more equations are given they may be taken in pairs, and the *same* unknown quantity eliminated from each pair.

2. Given  $x^m y^n = a, \quad (1)$  } Find  $x$  and  $y$  by the second  
 $x^p y^q = b, \quad (2)$  } method.

$$\text{From (1) } x^m = \frac{a}{y^n}; \quad \therefore x = \frac{a^{\frac{1}{m}}}{y^{\frac{n}{m}}};$$

$$\dots (2) \quad x^p = \frac{b}{y^q}; \quad \therefore x = \frac{b^{\frac{1}{p}}}{y^{\frac{q}{p}}}.$$

Hence, having two distinct values of the same quantity  $x$ , these values are equal to each other, and we may equate them; thus,

$$\frac{a^{\frac{1}{m}}}{y^{\frac{n}{m}}} = \frac{b^{\frac{1}{p}}}{y^{\frac{q}{p}}}, \quad \therefore \frac{y^{\frac{q}{p}}}{y^{\frac{n}{m}}} = \frac{b^{\frac{1}{p}}}{a^{\frac{1}{m}}}, \quad \text{or } y^{\frac{q}{p} - \frac{n}{m}} = \frac{b^{\frac{1}{p}}}{a^{\frac{1}{m}}},$$

$$\text{or, } y^{\frac{mq - np}{mp}} = \frac{b^{\frac{1}{p}}}{a^{\frac{1}{m}}}, \quad \therefore y = \left( \frac{b^{\frac{1}{p}}}{a^{\frac{1}{m}}} \right)^{\frac{mp}{mq - np}} = \left( \frac{b^m}{a^p} \right)^{\frac{1}{mq - np}}.$$

\* When the signs of the quantity to be eliminated are unlike, we must of course *add* the two equations.

And proceeding in a similar manner to obtain two distinct values of  $y$  from the given equations, and equate them, we shall have

$$x = \left( \frac{a^q}{b^n} \right)^{\frac{1}{mq - np}}.$$

$$3. \text{ Given } \left. \begin{array}{l} \frac{y}{x} + \frac{3x}{x+y} = \frac{x^2 - y^2}{y}, \quad (1) \\ \text{and } \frac{x}{y} = \frac{x+y}{x} + \frac{y}{x}, \quad \dots \quad (2) \end{array} \right\} \begin{array}{l} \text{Find } x \text{ and } y \text{ by the} \\ \text{third method.} \end{array}$$

$$(2) \quad \frac{x}{y} - \frac{y}{x} = \frac{x+y}{x}, \quad \frac{x^2 - y^2}{xy} = \frac{x+y}{x},$$

and dividing each member by  $\frac{x+y}{x}$ , we have

$$\frac{x-y}{y} = 1, \quad \therefore x-y = y,$$

$$\left. \begin{array}{l} x = 2y \\ x^2 = 4y^2 \end{array} \right\} \text{substituting these values in (1) we have}$$

$$\frac{y}{2y} + \frac{6y}{2y+y} = \frac{4y^2 - y^2}{y},$$

$$\frac{1}{2} + \frac{6}{2+1} = 4y - y, \quad \frac{1}{2} + \frac{6}{3} = 3y,$$

$$3 + 12 = 18y, \quad \text{or } 18y = 15,$$

$$\therefore y = \frac{15}{18} = \frac{5}{6}, \quad \text{and } x = 2y = \frac{5}{3}.$$

$$4. \text{ Given } \frac{xy}{x+y} = 70, \quad \frac{xz}{x+z} = 84, \quad \frac{yz}{y+z} = 140;$$

find  $x$ ,  $y$ , and  $z$ .

From the first equation, by taking the reciprocal\* of each

\* The reciprocal of any quantity is unity divided by that quantity; thus, the reciprocals of  $x$ ,  $\frac{x}{c}$ , and  $\frac{a+b}{x}$  are  $\frac{1}{x}$ ,  $\frac{c}{x}$ , and  $\frac{x}{a+b}$  respectively.

member, we have  $\frac{x+y}{xy} = \frac{1}{70}$ , or  $\frac{x}{xy} + \frac{y}{xy} = \frac{1}{70}$ ;

$$\therefore \frac{1}{x} + \frac{1}{y} = \frac{1}{70}, (1); \quad \text{and similarly from the second and third.}$$

$$\frac{1}{x} + \frac{1}{z} = \frac{1}{84}, (2)$$

$$\frac{1}{y} + \frac{1}{z} = \frac{1}{140}, (3)$$

$$(1) \quad \frac{1}{x} + \frac{1}{y} = \frac{2}{140}, \text{ and subtracting (3) from this,}$$

$$\frac{1}{x} - \frac{1}{z} = \frac{1}{140} = \frac{3}{420},$$

$$(2) \quad \frac{1}{x} + \frac{1}{z} = \frac{5}{420},$$

$$\therefore \frac{2}{x} = \frac{8}{420} = \frac{2}{105}, \therefore x = 105.$$

$$\text{and} \quad \frac{2}{z} = \frac{2}{420}; \quad \therefore z = 420.$$

$$(3) \quad \frac{1}{y} + \frac{1}{420} = \frac{1}{140}; \therefore \frac{1}{y} = \frac{3}{420} - \frac{1}{420} = \frac{2}{420} = \frac{1}{210};$$

$$\therefore y = 210.$$

*Solve the following Simple Equations.*

$$1. \sqrt{x^2 + 3} = \sqrt{7}. \quad \text{Ans. } x = 2.$$

$$2. \sqrt{x^2 - 16} = x - 2. \quad \text{Ans. } x = 5.$$

$$3. \sqrt{x} - 1 = \sqrt{x - 9}. \quad \text{Ans. } x = 25.$$

$$4. \sqrt{x} + \sqrt{x - 3} = 3. \quad \text{Ans. } x = 4.$$

$$5. \sqrt{x} - \sqrt{2} = \sqrt{x - 2}. \quad \text{Ans. } x = 2.$$

$$6. \sqrt[3]{4x + 3} = 3. \quad \text{Ans. } x = 6.$$



And proceeding in a similar manner to obtain two distinct values of  $y$  from the given equations, and equate them, we shall have

$$x = \left( \frac{a^q}{b^n} \right)^{\frac{1}{mq - np}}.$$

$$3. \text{ Given } \left. \begin{aligned} \frac{y}{x} + \frac{3x}{x+y} &= \frac{x^2 - y^2}{y}, & (1) \\ \text{and } \frac{x}{y} &= \frac{x+y}{x} + \frac{y}{x}, & \dots (2) \end{aligned} \right\} \begin{array}{l} \text{Find } x \text{ and } y \text{ by the} \\ \text{third method.} \end{array}$$

$$(2) \quad \frac{x}{y} - \frac{y}{x} = \frac{x+y}{x}, \quad \frac{x^2 - y^2}{xy} = \frac{x+y}{x},$$

and dividing each member by  $\frac{x+y}{x}$ , we have

$$\frac{x-y}{y} = 1, \quad \therefore x-y = y,$$

$$\left. \begin{aligned} x &= 2y \\ x^2 &= 4y^2 \end{aligned} \right\} \text{ substituting these values in (1) we have}$$

$$\frac{y}{2y} + \frac{6y}{2y+y} = \frac{4y^2 - y^2}{y},$$

$$\frac{1}{2} + \frac{6}{2+1} = 4y - y, \quad \frac{1}{2} + \frac{6}{3} = 3y,$$

$$3 + 12 = 18y, \quad \text{or } 18y = 15,$$

$$\therefore y = \frac{15}{18} = \frac{5}{6}, \quad \text{and } x = 2y = \frac{5}{3}.$$

$$4. \text{ Given } \frac{xy}{x+y} = 70, \quad \frac{xz}{x+z} = 84, \quad \frac{yz}{y+z} = 140;$$

find  $x$ ,  $y$ , and  $z$ .

From the first equation, by taking the reciprocal\* of each

\* The reciprocal of any quantity is unity divided by that quantity; thus, the reciprocals of  $x$ ,  $\frac{x}{c}$ , and  $\frac{a+b}{x}$  are  $\frac{1}{x}$ ,  $\frac{c}{x}$ , and  $\frac{x}{a+b}$  respectively.

member, we have  $\frac{x+y}{xy} = \frac{1}{70}$ , or  $\frac{x}{xy} + \frac{y}{xy} = \frac{1}{70}$ ;

$$\therefore \frac{1}{x} + \frac{1}{y} = \frac{1}{70}, (1); \text{ and similarly from the second and third.}$$

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$$(1) \quad \frac{1}{x} + \frac{1}{y} = \frac{2}{140}, \text{ and subtracting (3) from this,}$$

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$$\frac{1}{x} - \frac{1}{z} = \frac{1}{140} - \frac{1}{140} = \frac{0}{140},$$

$$(2) \quad \frac{1}{x} + \frac{1}{z} = \frac{5}{420},$$


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$$\therefore \frac{2}{x} = \frac{8}{420} = \frac{2}{105}, \therefore x = 105.$$

$$\text{and} \quad \frac{2}{x} = \frac{2}{420}; \quad \therefore x = 420.$$

$$(3) \quad \frac{1}{y} + \frac{1}{420} = \frac{1}{140}; \therefore \frac{1}{y} = \frac{3}{420} - \frac{1}{420} = \frac{2}{420} = \frac{1}{210};$$

$$\therefore y = 210.$$

*Solve the following Simple Equations.*

- |  |                |
|--|----------------|
| 1. $\sqrt{x^2 + 3} = \sqrt{7}.$          | Ans. $x = 2.$  |
| 2. $\sqrt{x^2 - 16} = x - 2.$            | Ans. $x = 5.$  |
| 3. $\sqrt{x} - 1 = \sqrt{x - 9}.$        | Ans. $x = 25.$ |
| 4. $\sqrt{x} + \sqrt{x - 3} = 3.$        | Ans. $x = 4.$  |
| 5. $\sqrt{x} - \sqrt{2} = \sqrt{x - 2}.$ | Ans. $x = 2.$  |
| 6. $\sqrt[3]{4x + 3} = 3.$               | Ans. $x = 6.$  |

$$7. \sqrt{5x+4} = \sqrt{3x} + 2. \quad \text{Ans. } x = 12.$$

$$8. \sqrt{x^2 + a^2} + x = b. \quad \text{Ans. } x = \frac{b^2 - a^2}{2b}.$$

$$9. 2\sqrt{x} - \sqrt{a} = 2\sqrt{x-a}. \quad \text{Ans. } x = \frac{25a}{16}.$$

$$10. a + x = \sqrt{x^2 + 5x - a}. \quad \text{Ans. } x = \frac{a^2 + a}{5 - 2a}.$$

$$11. \sqrt{a^2 + x^2} = \sqrt[4]{b^4 + x^4}. \quad \text{Ans. } x = \sqrt{\frac{b^4 - a^4}{2a^2}}.$$

$$12. \sqrt{a-x} = \frac{a}{\sqrt{a-x}} - x. \quad \text{Ans. } x = a - 1.$$

$$13. \sqrt{a+x} + \sqrt{a-x} = 2\sqrt{x}. \quad \text{Ans. } x = \frac{4}{5}a.$$

$$14. \frac{\sqrt{x}-2}{3} + 3 = \frac{x-4}{\sqrt{x}+2}. \quad \text{Ans. } x = 42\frac{1}{4}.$$

$$15. x - \sqrt{a^2 + x\sqrt{x^2-1}} = a. \quad \text{Ans. } x = \frac{4a^2+1}{4a}.$$

$$16. \frac{\sqrt{a+x} + \sqrt{a-x}}{\sqrt{a+x} - \sqrt{a-x}} = c. \quad \text{Ans. } x = \frac{2ac}{c^2+1}.$$

$$17. \sqrt{x+a} = \sqrt{a} + \sqrt{x-a}. \quad \text{Ans. } x = \frac{5a}{4}.$$

$$18. \sqrt{x} - \sqrt{a-x} = \frac{\sqrt{x} + \sqrt{a-x}}{2}. \quad \text{Ans. } x = \frac{9a}{10}.$$

$$19. \sqrt{\frac{a^2}{x} + b} - \sqrt{\frac{a^2}{x} - b} = c. \quad \text{Ans. } x = \frac{4a^2c^2}{4b^2+c^4}.$$

$$20. \sqrt{x^2+9} = 2 + \frac{x^2-9}{\sqrt{x^2+9}+3}. \quad \text{Ans. } x = 3\sqrt{15}.$$

$$21. \frac{na^2}{\sqrt{a^2+x^2}} - x = \sqrt{a^2+x^2}. \quad \text{Ans. } x = \frac{(n-1)a}{(2n-1)^{\frac{1}{2}}}.$$

$$22. ax - \sqrt{x^2 + x + 1} = \sqrt{x^2 - x + 1}.$$

$$\text{Ans. } x = \frac{2}{a} \sqrt{\frac{a^2 - 1}{a^2 - 4}}.$$

$$23. \sqrt{a^2 + ax} = a - \sqrt{a^2 - ax}. \quad \text{Ans. } x = \frac{a}{2} \sqrt{3}.$$

$$24. \frac{1}{x} + \frac{1}{a} = \sqrt{\frac{1}{a^2}} + \sqrt{\frac{4}{b^2 x^2} + \frac{1}{x^4}}.$$

$$\text{Ans. } x = \frac{ab^2}{a^2 - b^2}.$$

$$25. \frac{ax - 1}{\sqrt{ax} + 1} - 4 = \frac{\sqrt{ax} - 1}{2}. \quad \text{Ans. } x = \frac{81}{a}.$$

$$26. \frac{1}{2} \sqrt{x^2 + 3a^2} + \frac{1}{2} \sqrt{x^2 - 3a^2} = x \sqrt{a}.$$

$$\text{Ans. } x = \sqrt[4]{\frac{9a^3}{4 - 4a}}.$$

$$27. \frac{1}{\sqrt{1-x}+1} + \frac{1}{\sqrt{1+x}-1} = \frac{1}{x}. \quad \text{Ans. } x = \frac{\sqrt{3}}{2}.$$

$$28. \sqrt[3]{1+x} + \sqrt[3]{1-x} = \sqrt[3]{2}. \quad \text{Ans. } x = 1.$$

$$29. \frac{x + \sqrt{x^2 - 1}}{x - \sqrt{x^2 - 1}} + \frac{x - \sqrt{x^2 - 1}}{x + \sqrt{x^2 - 1}} = 4x(x - 1).$$

$$\text{Ans. } x = \frac{1}{2}.$$

$$30. \left. \begin{array}{l} \text{Given } x + y = 5 \\ \text{and } x - y = 1 \end{array} \right\} \text{ find } x \text{ and } y. \quad \begin{array}{l} \text{Ans. } x = 3. \\ y = 2. \end{array}$$

$$31. \left. \begin{array}{l} 2x - y = 1. \\ x + 3y = 11. \end{array} \right\} \quad \begin{array}{l} \text{Ans. } x = 2. \\ y = 3. \end{array}$$

$$32. \left. \begin{array}{l} 4x - 11y = 9. \\ 2x + 3y = 13. \end{array} \right\} \quad \begin{array}{l} \text{Ans. } x = 5. \\ y = 1. \end{array}$$

$$33. \left. \begin{array}{l} 3x + 2y = 23. \\ 5y - 2x = 29. \end{array} \right\} \quad \begin{array}{l} \text{Ans. } x = 3. \\ y = 7. \end{array}$$

*Problems.*

1. If to twice a certain number we add 9, the square root of the sum will be 5; what is the number? Ans. 8.

2. If to five times a certain number we add 4, the square root of the sum will be equal to 2 + the square root of thrice the number; find it. Ans. 12.

3. There is a number, from the square of which if 7 be subtracted, the square root of the remainder will be equal to 7 diminished by the number itself; what is the number? Ans. 4.

4. The sum of two numbers is 56, and their difference is 24; find them. Ans. 40 and 16.

5. The sum of two numbers is 16, and the sum of their reciprocals = twice the difference of their reciprocals; find the numbers. Ans. 4 and 12.

6. In a naval engagement, one-third of the fleet was taken, one-sixth sunk, and two ships were burnt; one-seventh of the remainder was lost in a storm after the action, and 24 ships remain; how many ships were in the fleet at first? Ans. 60.

7. The ages of two persons are in the ratio of 3 to 4, but 10 years ago the ratio of their ages was that of 2 to 3; what are their ages? Ans. 30 and 40.

8. A wine merchant mixed 20 gallons of spirits at 9s. a gallon with 36 gallons at 11s. a gallon, and he now wishes to add as many gallons at 14s. a gallon as will make the mixture worth 12s. a gallon; how many gallons of this last must he add? Ans. 48 gallons.

9. A grazier buys 12 sheep and 20 lambs of one person for 29*l.*; and 10 sheep and 30 lambs of another at the same price for 38*l.* 10*s.*; what was the price of each? Ans. Sheep, 25*s.*, lambs, 14*s.*

10. A and B made a joint stock of 833*l.*, which, after a successful speculation, produced a clear gain of 153*l.* Of this B had 45*l.* more than A. What did each person contribute to the stock? Ans. A 294*l.*; and B 539*l.*

11. Suppose that for every 10 sheep a farmer kept, he should plough an acre of ground, and be allowed one acre of pasture for every 4 sheep; how many sheep may that person keep who farms 700 acres? Ans. 2000.

12. If the numerator of a certain fraction be added to the

denominator, the result will be equal to 5 times the numerator; what is the fraction? Ans.  $\frac{1}{4}$ .

13. What number is that to which, if 1, 5, and 13 be severally added, the first sum shall be to the second as the second to the third? Ans. 8.

14. A sets out from C to go to D, at the same time that B sets out from D to go to C; A arrives at D  $a$  hours, and B at C  $b$  hours, after they met; how long did each take to perform the journey? Ans. A  $a^{\frac{1}{2}}(a^{\frac{1}{2}} + b^{\frac{1}{2}})$ .

B  $b^{\frac{1}{2}}(a^{\frac{1}{2}} + b^{\frac{1}{2}})$ .

15. A person has two sorts of wine, one worth 20 pence a quart, and the other 12 pence, from which he would mix a quart to be worth 14 pence; how much of each must he take? Ans.  $\frac{1}{4}$  of 1st;  $\frac{3}{4}$  of 2nd.

16. A number consisting of two digits is equal to 5 times the sum of its digits, and if the sum of its digits be added to the number, it will be inverted; what is the number? Ans. 45.

17. A person at play won twice as much as he began with, and then lost 16 shillings. After this he lost four-fifths of what remained, and then won as much as he began with, and counting his money, found he had 80 shillings; what sum did he begin with? Ans. 52 shillings.

18. A can perform a piece of work in  $a$  days, and B in  $b$  days; in how many days can they finish it, if they both work at it together? Ans.  $\frac{ab}{a+b}$  days.

19. A trader maintained himself for 3 years at the expense of 50*l*. a year, and in each of those years augmented that part of his stock which was not so expended by  $\frac{1}{3}$  thereof. At the end of the third year his original stock was doubled; what was that stock? Ans. 740*l*.

20. A man rows a boat, with the tide, 5 miles in  $\frac{3}{4}$  of an hour, and returns against a tide half as strong, in  $1\frac{1}{2}$  hour; required the velocity of the strongest tide. Ans.  $2\frac{2}{3}$  miles per hour.

21. Divide the number 11520 into 3 such parts that 9 times the sum of the first and second shall be equal to 7 times the sum of the second and third, and if the first be

subtracted from the second, 8 times the remainder shall be equal to the sum of the first and third.

Ans. 2880, 3840, and 4800.

22. A person bought a certain number of sheep for 94*l*; having lost 7 of them, he sold one-fourth of the remainder at prime cost for 20*l*: how many sheep had he at first?

Ans. 47.

23. A farmer pays his rent in certain fixed numbers of quarters of wheat and barley; when wheat is at 55*s*. and barley at 33*s*. a quarter, the portions of rent by wheat and barley are equal to one another; but when wheat is at 65*s*. and barley at 41*s*. a quarter, the rent is increased 7*l*: what is the corn-rent?

Ans. 6 qrs. wheat, 10 qrs. barley.

24. The circumference of the fore-wheel of a carriage is *a* feet, and that of the hind wheel *b* feet; what is the distance travelled by the carriage when the fore-wheel has made *n* revolutions more than the hind-wheel?

Ans.  $\frac{abn}{b-a}$  feet.

## CHAPTER VI.

### QUADRATIC EQUATIONS.

XVI. A *quadratic* equation is one in which the unknown quantity is involved to the *second power*, or *square*. If the *square* alone is contained in the equation, it is called a *pure quadratic*; but if the *square* and *first power* are both contained in the equation, it is called an *affected quadratic*. Thus,  $x^2 = 4$ , and  $ay^2 - a = b$ , are *pure quadratics*; and  $x^2 + 2x = 8$ ,  $y^2 - ay - c = 0$ , are *affected quadratics*.

Ex. 1. Solve the pure quadratic  $7x^2 + 18 = 4x^2 + 450$ .

By transposition,  $7x^2 - 4x^2 = 450 - 18$ ,

$$\text{or } 3x^2 = 432,$$

$$\therefore x^2 = \frac{432}{3} = 144,$$

and extracting the square root of each member, we have

$$x = \pm 12.$$

The sign  $\pm$  is prefixed to the value of *x* because

$$+12 \times +12 = +144, \text{ and } -12 \times -12 = +144.$$

*Examples.*

Solve the following pure quadratics.

$$1. x^2 = 144. \quad \text{Ans. } x = \pm 12.$$

$$2. x^2 - 9 = 16. \quad \text{Ans. } x = \pm 5.$$

$$3. \frac{3x^2}{4} - 5 = 7. \quad \text{Ans. } x = \pm 4.$$

$$4. 2\sqrt{1-x^2} = \sqrt{3}. \quad \text{Ans. } x = \pm \frac{1}{2}.$$

$$5. \frac{\sqrt{a^2 + x^2} - x}{x} = \frac{1}{b}. \quad \text{Ans. } x = \frac{ab}{\sqrt{2b+1}}.$$

$$6. \frac{7x^2}{4} - \frac{4x^2 + 5}{2} + \frac{2x^2 - 15}{4} = 0. \quad \text{Ans. } x = \pm 5.$$

$$7. \frac{1}{2x^2} + 7 = \frac{9}{4x^2}. \quad \text{Ans. } x = \pm \frac{1}{2}.$$

$$8. 35 - \frac{x^2 + 50}{5} = x^2 - \frac{x^2 - 10}{3}. \quad \text{Ans. } x = \pm 5.$$

## ADFFECTED QUADRATICS.

XVII. Any adfected quadratic may be reduced to the form  $x^2 + px = q$ , where  $p$  and  $q$  represent known quantities or numbers; and if we add  $\left(\frac{p}{2}\right)^2$  to each side of this equation, we shall have

$$x^2 + px + \frac{p^2}{4} = \frac{p^2}{4} + q;$$

and taking the square root of each side, we have

$$x + \frac{p}{2} = \pm \sqrt{\frac{p^2}{4} + q}; \quad \therefore x = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} + q}.$$

Hence this rule for the solution of an adfected quadratic.  
 "Add the *square of half the coefficient of the second term* to each side, and then extract the square root of each side."



*Examples.*

- 1.
- $x^2 + 8x = 20$
- , Complete the square.

$$x^2 + 8x + 4^2 = 20 + 16 = 36, \text{ Extract the root.}$$

$$x + 4 = \pm 6;$$

$$\therefore x = \pm 6 - 4 = +6 - 4, \text{ or } -6 - 4 = 2, \text{ or } -10$$

- 2.
- $x^2 - 5x = 6$
- , Complete the square.

$$x^2 - 5x + \left(\frac{5}{2}\right)^2 = 6 + \frac{25}{4} = \frac{24}{4} + \frac{25}{4} = \frac{49}{4}, \text{ Extract the root,}$$

$$x - \frac{5}{2} = \pm \frac{7}{2},$$

$$\therefore x = \frac{5 \pm 7}{2} = \frac{12}{2}, \text{ or } -\frac{2}{2} = 6, \text{ or } -1.$$

It may be seen by these examples, that an easy way of taking the square root of the first side is to take the square roots of the first and third terms, and place between them the sign of the second term.

It also appears that a quadratic equation has two roots, or values of the unknown quantity.

Let the two roots of the equation  $x^2 + px - q = 0$ , be represented by  $\alpha$  and  $\beta$  respectively, then

$$\alpha = -\frac{p}{2} + \sqrt{\frac{p^2}{4} + q}, \quad \beta = -\frac{p}{2} - \sqrt{\frac{p^2}{4} + q}.$$

$$\therefore \alpha + \beta = -p, \quad \alpha\beta = \frac{p^2}{4} - \left(\frac{p^2}{4} + q\right) = -q;$$

that is, the *sum of the roots* = the coefficient of the second term with its sign changed, and the *product of the roots* = the last term. These propositions are true of equations of any degree.

It is obvious that the two roots above found are *equal* only when  $\sqrt{\frac{p^2}{4} + q} = 0$ , for then  $\alpha = -\frac{p}{2}$  and  $\beta = -\frac{p}{2}$ . When  $q$  is positive, both roots are *real*; but when  $q$  is negative, they are *real* only when  $\frac{p^2}{4} > q$ , for if  $\frac{p^2}{4} < q$  and

$p$  is negative,  $\frac{p^2}{4} - q$  is negative, and  $\sqrt{\frac{p^2}{4} - q}$  is *imaginary*, since a negative quantity has no square root.

Any equation of the form  $ax^{2n} + bx^n = c$ , where the index of the unknown quantity in the first term is *double* the index of the unknown quantity in the second term, may be solved as an affected quadratic.

Equations of a higher degree may frequently be made to assume the *form* of pure or affected quadratics, and may then be solved accordingly.

### Examples.

1.  $x^4 - 7x^2 = 8$ , Complete the square.

$$x^4 - 7x^2 + \frac{49}{4} = 8 + \frac{49}{4} = \frac{32}{4} + \frac{49}{4} = \frac{81}{4}, \text{ Extract the root.}$$

$$x^2 - \frac{7}{2} = \pm \frac{9}{2}, \quad x^2 = \frac{7 \pm 9}{2} = \frac{16}{2}, \text{ or } -\frac{2}{2} = 8, \text{ or } -1,$$

$$\therefore x = \pm \sqrt{8}, \text{ or } \pm \sqrt{-1} = \pm 2\sqrt{2}, \text{ or } \pm \sqrt{-1}.$$

This equation being a *biquadratic*, has four roots, namely,  $+2\sqrt{2}$ ,  $-2\sqrt{2}$ ,  $+\sqrt{-1}$ , and  $-\sqrt{-1}$ ; the first two *real*, and the remaining two *imaginary*.

$$2. \frac{1}{3} \sqrt{x^2 + \frac{x}{2} + \frac{17}{2}} + \frac{2x^2 + x}{4} = 15\frac{3}{4},$$

$$\frac{1}{3} \sqrt{x^2 + \frac{x}{2} + \frac{17}{2}} + \frac{x^2}{2} + \frac{x}{4} = \frac{63}{4}, \text{ Multiply by 2.}$$

$$\frac{2}{3} \sqrt{x^2 + \frac{x}{2} + \frac{17}{2}} + x^2 + \frac{x}{2} = \frac{63}{2}, \quad \text{Add } \frac{17}{2} \text{ to}$$

each side.

$$x^2 + \frac{x}{2} + \frac{17}{2} + \frac{2}{3} \sqrt{x^2 + \frac{x}{2} + \frac{17}{2}} = 40,$$

Now,  $\therefore \left(x^2 + \frac{x}{2} + \frac{17}{2}\right)$  is the square of

$\sqrt{x^2 + \frac{x}{2} + \frac{17}{2}}$ , the equation has assumed the adfective form, and we may complete the square;

$$\left(x^2 + \frac{x}{2} + \frac{17}{2}\right) + \frac{2}{9} \sqrt{x^2 + \frac{x}{2} + \frac{17}{2} + \frac{1}{9}}^2 = \frac{361}{9},$$

$$\sqrt{x^2 + \frac{x}{2} + \frac{17}{2} + \frac{1}{9}} = \pm \frac{19}{3},$$

$$\sqrt{x^2 + \frac{x}{2} + \frac{17}{2}} = \frac{\pm 19 - 1}{3} = \frac{18}{3} = 6, \text{ or } -\frac{20}{3}.$$

Squaring both sides of this equation, we have

$$x^2 + \frac{x}{2} + \frac{17}{2} = 36 \text{ or } \frac{400}{9}, \text{ and, taking the value 36,}$$

$$x^2 + \frac{1}{2}x = 36 - \frac{17}{2} = \frac{72 - 17}{2} = \frac{55}{2}.$$

$$x^2 + \frac{1}{2}x + \frac{1}{4} = \frac{55}{2} + \frac{1}{16} = \frac{440}{16} + \frac{1}{16} = \frac{441}{16},$$

$$x + \frac{1}{4} = \pm \frac{21}{4},$$

$$\therefore x = \frac{\pm 21 - 1}{4} = \frac{20}{4} \text{ or } -\frac{22}{4} = 5, \text{ or } -\frac{11}{2}.$$

$$\text{Again, } x^2 + \frac{1}{2}x = \frac{400}{9} - \frac{17}{2} = \frac{800 - 153}{18} = \frac{647}{18},$$

$$x^2 + \frac{1}{2}x + \frac{1}{4} = \frac{647}{18} + \frac{1}{16} = \frac{5185}{144},$$

$$x + \frac{1}{4} = \pm \frac{\sqrt{5185}}{12}, \therefore x = -\frac{1}{4} \pm \frac{\sqrt{5185}}{12}.$$

$$\text{Hence } x = 5 \text{ or } -\frac{11}{2} \text{ or } -\frac{3 \pm \sqrt{5185}}{12}.$$

3.  $x^3 - 3x = 2$ , Multiply by  $x$ .

$$x^4 - 3x^2 = 2x, \quad x^4 - 2x^2 = x^2 + 2x,$$

$$x^4 - 2x^2 + 1 = x^2 + 2x + 1,$$

$$x^2 - 1 = x + 1, \quad x^2 - x = 2,$$

$$x^2 - x + \frac{1}{2} = 2 + \frac{1}{4} = \frac{9}{4}, \quad x - \frac{1}{2} = \pm \frac{3}{2},$$

$$\therefore x = \frac{1 \pm 3}{2} = \frac{4}{2}; \text{ or } -\frac{2}{2} = 2; \text{ or } -1.$$

4.  $x^3 - 2x = 4$ ,

$$x^4 - 2x^2 = 4x, \text{ Add } 4x^2 \text{ to each side.}$$

$$x^4 + 2x^2 = 4x^2 + 4x,$$

$$x^4 + 2x^2 + 1 = 4x^2 + 4x + 1,$$

$$x^2 + 1 = \pm (2x + 1), \quad x^2 = 2x, \quad \therefore x = 2.$$

$$x^2 + 1 = -2x - 1, \quad x^2 + 2x + 1 = -1,$$

$$x + 1 = \pm \sqrt{-1}, \quad \therefore x = -1 \pm \sqrt{-1}.$$

5.  $(x^2 - 5)^2 = (x - 3)^2 + (x + 1)^2$ ,

$$x^4 - 10x^2 + 25 = x^2 - 6x + 9 + x^2 + 2x + 1,$$

$$x^4 - 12x^2 + 4x + 15 = 0,$$

$$x^4 - 8x^2 = 4x^2 - 4x - 15,$$

$$x^4 - 8x^2 + 16 = 4x^2 - 4x + 1. \text{ Extract the root.}$$

$$x^2 - 4 = \pm (2x - 1), \quad x^2 - 2x = 3,$$

$$x^2 - 2x + 1 = 4, \quad x - 1 = \pm 2,$$

$$\therefore x = 1 \pm 2 = 3, \text{ or } -1.$$

Again,  $x^2 - 4 = -2x + 1, \quad x^2 + 2x = 5,$

$$x^2 + 2x + 1 = 6, \quad x + 1 = \pm \sqrt{6},$$

$$\therefore x = \pm \sqrt{6} - 1.$$

6.  $x^3 - x^2 = 4$ , Multiply by 4.

$4x^3 = 4x^3 + 16$ , Add  $x^4$  to each side.

$x^4 + 4x^3 + 4x^2 = x^4 + 8x^2 + 16$ , Extract the root.

$$x^2 + 2x = x^2 + 4,$$

$$2x = 4, \therefore x = 2.$$

7.  $(x-1)\sqrt{2x-x^2} = \frac{1}{2}$ ,

$$(x^2 - 2x + 1)(2x - x^2) = \frac{1}{4},$$

$$-x^4 + 4x^3 - 5x^2 + 2x = \frac{1}{4},$$

$$4x^4 - 16x^3 + 20x^2 - 8x = -1,$$

$$4x^4 - 16x^3 + 16x^2 + 4x^2 - 8x = -1,$$

$$(2x^2 - 4x)^2 + 2(2x^2 - 4x) + 1 = 0, \text{ Extract the root.}$$

$$2x^2 - 4x + 1 = 0, \quad x^2 - 2x = -\frac{1}{2},$$

$$x^2 - 2x + 1 = 1 - \frac{1}{2} = \frac{1}{2}, \quad x - 1 = \pm \frac{1}{\sqrt{2}}$$

$$\therefore x = 1 \pm \frac{1}{\sqrt{2}} = \frac{\sqrt{2} \pm 1}{\sqrt{2}}.$$

8.  $x - 2\sqrt{x+2} = 1 + \sqrt[4]{x^2 - 3x + 2}$ ,

$$\begin{aligned} (x-1) - 2\sqrt{x+2} &= \sqrt[4]{(x-1)^2(x+2)} \\ &= \sqrt{x-1} \cdot \sqrt[4]{x+2}, \end{aligned}$$

$$(x-1) - \sqrt[4]{x+2} \cdot \sqrt{x-1} = 2\sqrt{x+2},$$

$$(x-1) - \sqrt[4]{x+2} \cdot \sqrt{x-1} + \frac{1}{4}\sqrt{x+2} =$$

$$2\sqrt{x+2} + \frac{1}{4}\sqrt{x+2} = \frac{9}{4}\sqrt{x+2},$$

$$\sqrt{x-1} - \frac{1}{2}\sqrt[4]{x+2} = \pm \frac{3}{2}\sqrt[4]{x+2},$$

$$\sqrt{x-1} = 2\sqrt[4]{x+2}, \quad x-1 = 4\sqrt{x+2},$$

$$x^2 - 2x + 1 = 16x + 32, \quad x^2 - 18x = 31,$$

$$x^2 - 18x + 9^2 = 81 + 31 = 112 = 16 \times 7,$$

$$x - 9 = \pm 4\sqrt{7}, \quad \therefore x = 9 \pm 4\sqrt{7}.$$

Again,  $\sqrt{x-1} = -\sqrt{x+2}$ ,

$$x^2 - 2x + 1 = x + 2, \quad x^2 - 3x = 1,$$

$$x^2 - 3x + \frac{9}{2} = 1 + \frac{9}{2} = \frac{13}{2}, \quad x - \frac{3}{2} = \pm \frac{\sqrt{13}}{2},$$

$$\therefore x = \frac{3 \pm \sqrt{13}}{2} = \frac{1}{2} (3 \pm \sqrt{13}).$$

9.  $\sqrt{a^2 - x^2} + x \sqrt{a^2 - 1} = a^2 \sqrt{1 - x^2},$

$$a^2 - x^2 + x^2(a^2 - 1) + 2x\{(a^2 - x^2)(a^2 - 1)\}^{\frac{1}{2}} = a^4(1 - x^2),$$

$$a^2 - a^2 + a^2x^2 - a^2 + 2x\{a^4 - a^2x^2 - a^2 + x^2\}^{\frac{1}{2}} = a^4 - a^4x^2,$$

$$a^4 - a^2x^2 - a^2 + x^2 - 2x\sqrt{a^4 - a^2x^2 - a^2 + x^2} + x^2 = a^4x^2,$$

$$\sqrt{a^4 - a^2x^2 - a^2 + x^2} = a^2x + x$$

$$a^4 - a^2x^2 - a^2 + x^2 = a^4x^2 + 2a^2x^2 + x^2,$$

$$a^4 - a^2 = a^4x^2 + 3a^2x^2, \quad a^2 - 1 = a^2x^2 + 3x^2,$$

$$a^2 + 3)x^2 = a^2 - 1, \quad x^2 = \frac{a^2 - 1}{a^2 + 3}, \quad \therefore x = \sqrt{\frac{a^2 - 1}{a^2 + 3}}.$$

10.  $\sqrt{x - \frac{1}{x}} + \sqrt{1 - \frac{1}{x}} = x,$

$$\sqrt{1 - \frac{1}{x}} = x - \sqrt{x - \frac{1}{x}},$$

$$1 - \frac{1}{x} = x^2 - 2x\sqrt{x - \frac{1}{x}} + x - \frac{1}{x},$$

$$x^2 - 1 - 2x\frac{\sqrt{x^2 - 1}}{\sqrt{x}} + x = 0,$$

$$x^2 - 1 - 2x^{\frac{1}{2}}\sqrt{x^2 - 1} + x = 0,$$

$$\sqrt{x^2 - 1} - x^{\frac{1}{2}} = 0, \quad \sqrt{x^2 - 1} = x^{\frac{1}{2}}, \quad x^2 - 1 = x,$$

$$x^2 - x = 1, \quad x^2 - x + \frac{1}{2} = 1 + \frac{1}{2} = \frac{3}{2},$$

$$x - \frac{1}{2} = \frac{\sqrt{5}}{2}, \quad \therefore x = \frac{1 \pm \sqrt{5}}{2}.$$

$$11. \quad \left(\frac{1+x}{1-x}\right)^{\frac{1}{4}} + \sqrt{\frac{1-a}{1+a}} \sqrt[4]{\frac{1-x}{1+a}} = 2 \sqrt[4]{\frac{1-a^2}{(1+a)^2}}.$$

$$\frac{\sqrt[4]{1+x}}{\sqrt[4]{1-x}} + \sqrt{\frac{1-a}{1+a}} \frac{\sqrt[4]{1-x}}{\sqrt[4]{1+a}} = 2 \sqrt[4]{\frac{1-a}{1+a}} \times \frac{\sqrt[4]{1+x}}{\sqrt[4]{1-x}}$$

$$\frac{\sqrt{1+x}}{\sqrt{1-x}} + \sqrt{\frac{1-a}{1+a}} = 2 \sqrt[4]{\frac{1-a}{1+a}} \cdot \frac{\sqrt[4]{1+x}}{\sqrt[4]{1-x}},$$

$$\sqrt{\frac{1+x}{1-x}} - 2 \sqrt[4]{\frac{1-a}{1+a}} \cdot \sqrt[4]{\frac{1+x}{1-x}} + \sqrt{\frac{1-a}{1+a}} = 0,$$

$$\sqrt[4]{\frac{1+x}{1-x}} - \sqrt[4]{\frac{1-a}{1+a}} = 0,$$

$$\sqrt[4]{\frac{1+x}{1-x}} = \sqrt[4]{\frac{1-a}{1+a}},$$

$$\frac{1+x}{1-x} = \frac{1-a}{1+a}, \quad \therefore x = -a.$$

$$12. \quad \frac{1+x^4}{(1+x)^4} = \frac{1}{2}.$$

$$2 + 2x^4 = 1 + 4x + 6x^2 + 4x^3 + x^4,$$

$$x^4 - 4x^3 - 6x^2 - 4x + 1 = 0 \dots\dots (1). \quad \text{Divide by } x^2$$

$$x^2 - 4x - 6 - \frac{4}{x} + \frac{1}{x^2} = 0,$$

$$x^2 + \frac{1}{x^2} - 4x - \frac{4}{x} = 6, \quad \text{Add 2 to each side.}$$

$$x^2 + 2 + \frac{1}{x^2} - 4\left(x + \frac{1}{x}\right) = 8,$$

$$\left(x + \frac{1}{x}\right)^2 - 4\left(x + \frac{1}{x}\right) + 2^2 = 8 + 4 = 12,$$

$$x + \frac{1}{x} - 2 = \pm 2\sqrt{3},$$

$$x^2 + 1 - 2x = \pm 2\sqrt{3} \cdot x,$$

$$x^2 - (2 \pm 2\sqrt{3})x = -1,$$

$$x^2 - 2(1 \pm \sqrt{3})x + (1 \pm \sqrt{3})^2 = 1 \pm 2\sqrt{3} + 3 - 1 \\ = 3 \pm 2\sqrt{3},$$

$$x - (1 \pm \sqrt{3}) = \pm \sqrt{3 \mp 2\sqrt{3}};$$

$$\therefore x = 1 \pm \sqrt{3} \pm \sqrt{3 \mp 2\sqrt{3}}.$$

$$13. \quad 2x^2 - x^3 = 1, \quad 4x^4 - x^2 = 2x + 1,$$

$$2x^2 = x + 1, \quad x^2 - \frac{x}{2} = \frac{1}{2},$$

$$x^2 - \frac{x}{2} + \frac{1}{16} = \frac{1}{16} + \frac{8}{16} = \frac{9}{16}, \quad x - \frac{1}{4} = \frac{3}{4},$$

$$\therefore x = \frac{7}{4} = 1.$$

This method will apply to any equation of the form  $ax^3 \pm bx^2 = c$ ; that is, we may solve the equation by multiplying by  $ax \mp b$ , using the upper sign when the coefficient of  $x^2$  is positive, and the lower when it is negative.

But as this method introduces a factor which does not belong to the original equation, we must take care not to put this factor equal to 0, for this would give a value of  $x$ , which would not satisfy the conditions of the equation.

*Solve the following Adfected Quadratics.*

1. Given  $x^2 - 8x = 9$ , to find the values of  $x$ .

Ans.  $x = 9$ , or  $-1$ .

2.  $x^2 + 12x - 16 = 92$ .

Ans.  $x = 6$ , or  $-18$ .

3.  $x^2 - 3x = 10$ .

Ans.  $x = 5$ , or  $-2$ .

4.  $x^2 - x + 3 = 45$ .

Ans.  $x = 7$ , or  $-6$ .

5.  $5x^2 + x = 4$ .

Ans.  $x = \frac{4}{5}$ , or  $-1$ .



6.  $2x^2 - x = 21$ .      Ans.  $x = \frac{7}{2}$ , or  $-3$ .
7.  $5x^2 + 6x - 3 = 60$ .      Ans.  $x = 3$ , or  $-\frac{21}{5}$ .
8.  $x - 1 = -\frac{1}{x}$ .      Ans.  $x = \frac{1 \pm \sqrt{-3}}{2}$ .
9.  $(x - 12)(x + 2) = 0$ .      Ans.  $x = 12$ , or  $-2$ .
10.  $3x^2 - 14x + 15 = 0$ .      Ans.  $x = 3$ , or  $1\frac{2}{3}$ .
11.  $2x^2 - 11x = 21$ .      Ans.  $x = 7$ , or  $-1\frac{1}{2}$ .
12.  $ax^2 - bx = c$ .      Ans.  $x = \frac{b \pm \sqrt{b^2 + 4ac}}{2a}$ .
13.  $4x - \frac{14 - x}{x + 1} = 14$ .      Ans.  $x = 4$ , or  $-\frac{7}{5}$ .
14.  $x^2 - 4ax = -7a^2$ .      Ans.  $x = (2 \pm \sqrt{-3})a$ .
15.  $\frac{10}{x} - \frac{14 - 2x}{x^2} = \frac{22}{9}$ .      Ans.  $x = 3$ , or  $\frac{21}{11}$ .
16.  $x + \sqrt{5x + 10} = 8$ .      Ans.  $x = 18$ , or  $3$ .
17.  $\frac{3x + 4}{5} - \frac{30 - 2x}{x - 6} = \frac{7x - 14}{10}$ .      Ans.  $x = 36$ , or  $12$ .
18.  $x + \sqrt{10x + 6} = 9$ .      Ans.  $x = 25$ , or  $3$ .
19.  $(x + 2)^2 = 2x^2 + 8$ .      Ans.  $x = 2$ .
20.  $\frac{x + 22}{3} - \frac{9x - 6}{2} = \frac{4}{x}$ .      Ans.  $x = 2$ , or  $\frac{12}{25}$ .
21.  $\frac{2x}{9} - 2 = \frac{3x - 16}{18} - \frac{4x - 3}{4x + 3}$ .      Ans.  $x = 6$ , or  $-4\frac{3}{4}$ .
22.  $\frac{x - 3}{x + 5} - \frac{x + 4}{x - 7} = 27$ .      Ans.  $x = 4$ , or  $8\frac{2}{3}$ .
23.  $x^2 - (a + b)x + ab = 0$ .      Ans.  $x = a$ , or  $b$ .
24.  $4x + 4\sqrt{x + 2} = 7$ .      Ans.  $x = 4\frac{1}{4}$ , or  $\frac{1}{4}$ .

$$25. x = \frac{x-9}{x^2+3} + 15. \quad \text{Ans. } x = 9, \text{ or } 16.$$

$$26. \sqrt{x+6} + \sqrt{x+3} = 3\sqrt{x}. \quad x = \frac{9 \pm \sqrt{76}}{5}.$$

$$27. \frac{\sqrt{4x+20}}{4+\sqrt{x}} = \frac{4-\sqrt{x}}{\sqrt{x}}. \quad \text{Ans. } x = 4, \text{ or } -\frac{64}{8}.$$

$$28. \sqrt{x+2} = \sqrt{7+2x}. \quad \text{Ans. } x = 9, \text{ or } 1.$$

$$29. \frac{x}{x+1} + \frac{x+1}{x} = 2\frac{1}{6}. \quad \text{Ans. } x = 2, \text{ or } -3.$$

$$30. \frac{4x^2}{3} = \frac{x}{3} + 11. \quad \text{Ans. } x = 3, \text{ or } -2\frac{3}{4}.$$

$$31. \sqrt{x-a} + \sqrt{x+b} = 2\sqrt{x}. \quad x = \frac{(a+b)^2}{8(b-a)}.$$

$$32. \frac{\sqrt{a^2x+b}}{a+\sqrt{x}} = \frac{a-\sqrt{x}}{\sqrt{x}}.$$

$$\text{Ans. } x = \frac{-(2a^2+b) \pm \sqrt{4a^6+4a^2b+b^2}}{2(a^2-1)}.$$

$$33. \frac{x+4}{3} - \frac{4x+7}{9} = \frac{7-x}{x-3} - 1. \quad \text{Ans. } x = 21, \text{ or } 5$$

$$34. \sqrt{5a+x} + \sqrt{5a-x} = \frac{12a}{\sqrt{5a+x}}. \quad \text{Ans. } x = 4a.$$

$$35. x^4 - 8x^2 = 9. \quad \text{Ans. } x = \pm 3, \text{ or } \pm \sqrt{-1}.$$

$$36. x^5 - 4x^3 = 32. \quad \text{Ans. } x = 2, \text{ or } \sqrt[3]{-4}.$$

$$37. \frac{nx+b}{\sqrt{x}} = \frac{na+b}{\sqrt{a}}. \quad \text{Ans. } x = a, \text{ or } \frac{b^2}{an^2}.$$

$$38. x^4 - 2x^2 = 3. \quad \text{Ans. } x = \pm \sqrt{3} \text{ or } \pm \sqrt{-1}.$$

$$39. \sqrt{4a+x} + \sqrt{a+x} = 2\sqrt{2a+x}. \quad x = -\frac{7}{8}a.$$

$$40. x^3 - x^3 = 56. \quad \text{Ans. } x = 4, \text{ or } \sqrt[3]{49}.$$

$$41. x + 5 = \sqrt{x+5} + 6. \quad \text{Ans. } x = 4, \text{ or } -1.$$

$$42. \sqrt{(2x+1)} + 2\sqrt{x} = \frac{21}{\sqrt{(2x+1)}}. \quad \text{Ans. } x = 4, \text{ or } -25.$$

$$43. x^3 + 20x^3 = 69. \quad \text{Ans. } x = \sqrt[3]{3}, \text{ or } \sqrt[3]{-23}$$

$$44. \frac{a+x}{\sqrt{a-x}} + \frac{a-x}{\sqrt{a+x}} = 2\sqrt{a}. \quad \text{Ans. } x = \pm a\sqrt{8\sqrt{2}-11}.$$

$$45. x\sqrt{\frac{a}{x}-1} = \sqrt{x^2-b^2}. \quad \text{Ans. } x = \frac{1}{4}(a \pm \sqrt{a^2+8b^2}).$$

$$46. (a+1)(x-1)^2 = 2(x^2+1). \quad \text{Ans. } x = \frac{\sqrt{a}+1}{\sqrt{a}-1}, \text{ or } \frac{\sqrt{a}-1}{\sqrt{a}+1}.$$

$$47. x^2 - 7x + \sqrt{x^2 - 7x + 18} = 24. \quad \text{Ans. } x = 9, \text{ or } -2, \text{ or } \frac{7 \pm \sqrt{173}}{2}$$

$$48. \left(\frac{1}{x}\right)^3 + a = \left(\frac{1}{x}\right)^{\frac{3}{2}}. \quad \text{Ans. } x = \left\{ \frac{1 \mp \sqrt{1-4a}}{2a} \right\}^{\frac{2}{3}}.$$

$$49. 5^x + \frac{125}{5^x} = 30. \quad \text{Ans. } x = 2, \text{ or } 4.$$

$$50. \sqrt{a+x} + \sqrt{a-x} = \frac{b}{\sqrt{a+x}}. \quad \text{Ans. } x = \frac{1}{2}(b-a \pm \sqrt{a^2+2ab-b^2}).$$

$$51. x^2 - 2x + 6\sqrt{x^2 - 2x + 5} = 11. \quad \text{Ans. } x = 1, \text{ or } 1 \pm 2\sqrt{13}.$$

$$52. x^2(x-1) = 8(x+2). \quad \text{Ans. } x = \frac{\sqrt{-7-3}}{2}$$

$$53. x^4 - 2x^2 + x = a. \quad \text{Ans. } x = \frac{1 \pm \sqrt{3 \pm 2\sqrt{4a+1}}}{2}$$

$$54. \frac{x}{\sqrt{x} + \sqrt{a-x}} + \frac{x}{\sqrt{x} - \sqrt{a-x}} = \frac{b}{\sqrt{x}}.$$

$$\text{Ans. } x = \frac{b \pm \sqrt{b^2 - 2ab}}{2}.$$

$$55. x^3 + \frac{1}{x\sqrt{x}} = 2.$$

$$\text{Ans. } x = 1, \text{ or } \frac{\sqrt[3]{2}}{2}.$$

$$56. ax - b^2 = x^2 - 2b\sqrt{a^2 - ax + x^2}.$$

$$\text{Ans. } x = \frac{a \pm \sqrt{a^2 \pm 8ab + 4b^2}}{2}.$$

$$57. 9x - 3x^2 + 4\sqrt{x^2 - 3x + 5} = 11.$$

$$\text{Ans. } x = \frac{3 \pm \sqrt{5}}{2}.$$

$$58. \sqrt{\frac{a}{x}} + \sqrt{\frac{x}{a}} = \frac{\sqrt{2ax - x^2}}{x}.$$

$$\text{Ans. } x = -\frac{1}{2}a(3 \mp \sqrt{13}).$$

$$59. x^2 + \frac{1}{x^2} + x + \frac{1}{x} = 4. \quad \text{Ans. } x = 1, \text{ or } \frac{-3 \pm \sqrt{5}}{2}.$$

$$60. x = \frac{12 + 8x^{\frac{1}{2}}}{x - 5}. \quad \text{Ans. } x = \frac{-3 \mp \sqrt{(-7)}}{2}.$$

$$61. \sqrt{x^2 + 1} - \sqrt{x^2 - 1} = \frac{x}{\sqrt{x^4 - 1}}.$$

$$\text{Ans. } x = \pm \sqrt{\frac{2}{\sqrt{3}}}.$$

$$62. x^3 - 6x - 4 = 5. \quad \text{Ans. } x = 3, \text{ or } \frac{-3 \pm \sqrt{-3}}{2}.$$

$$63. 3x^2 = \frac{2}{x} + \frac{13}{3}. \quad \text{Ans. } x = -\frac{2}{3}, \text{ or } \frac{1 \pm \sqrt{10}}{3}.$$

$$64. ax + 2\sqrt{n^2x + nax^2} = (3x - 1)n.$$

$$\text{Ans. } x = \frac{n}{n-a}, \text{ or } \frac{n}{9n-a}.$$

$$65. 2(\sqrt{1-x} + 1) = \frac{x}{\sqrt{1+x}-1}. \quad \text{Ans. } x = \frac{24}{25}.$$

$$66. (x-b)^{2n} + 2b^n x = b^{n+1}(2 + b^{n-1}) + 2(x-b)^{n+1}. \\ \text{Ans. } x = 2b.$$

$$67. \left(x - \frac{ab}{x}\right)^2 = \frac{a^2 + ab}{2} \cdot \left(\frac{a^2}{x^2} + 1\right). \\ \text{Ans. } x = \sqrt{a^2 + 2ab}, \text{ or } \sqrt{\frac{ab-a^2}{2}}.$$

$$68. x^4 = -1. \quad x = \sqrt[4]{-1}, \text{ or } \frac{\pm 1 \pm \sqrt{-1}}{\sqrt{2}}.$$

$$69. x^{4n} - 2x^{2n} + x^n = 6. \\ \text{Ans. } x = \sqrt[2n]{\frac{1}{2} \pm \frac{1}{2}\sqrt{13}}, \text{ or } \sqrt[2n]{\frac{1}{2} \pm \frac{1}{2}\sqrt{-7}}.$$

$$70. \{(x-2)^2 - x\}^2 - 90 + x = (x-2)^2. \\ \text{Ans. } x = 6, -1, \text{ or } \frac{5 \pm 3\sqrt{-3}}{2}.$$

$$71. \sqrt[n]{(1+x)^2} - \sqrt[n]{(1-x)^2} = \sqrt[n]{1-x^2}. \\ \text{Ans. } x = \frac{(1 \pm \sqrt{5})^n - 2^n}{(1 \pm \sqrt{5})^n + 2^n}.$$

$$72. 8x^4 + 4x^3 - 18x^2 + 11x - 2 = 0. \\ \text{Ans. } x = \frac{1}{2}, \text{ or } -2.$$

$$73. 4x^2 - 20 - 5\left(x + \frac{3}{x}\right) = -\frac{36}{x^2}. \\ \text{Ans. } x = 3, 1, \text{ or } \frac{-11 \pm \sqrt{-71}}{8}.$$

$$74. \frac{2}{(x-4)^{\frac{2}{3}}} + \frac{(x-4)^{\frac{1}{3}}}{2} = \frac{17}{4(x-4)^{\frac{1}{3}}}. \\ \text{Ans. } x = 12, \text{ or } 4\frac{1}{2}.$$

$$75. x^4 + 4x^3 + 3x^2 + 4x + 1 = 0.$$

$$\text{Ans. } x = \frac{-2 \pm \sqrt{3} \pm \sqrt{(3 \mp 4\sqrt{3})}}{2}.$$

$$76. (a^{2m} + 1)(x^{\frac{1}{2}} - 1)^2 = 2(x + 1).$$

$$\text{Ans. } x = \left( \frac{a^{2m} \pm 1}{a^{2m} \mp 1} \right)^2.$$

$$77. (x^2 - 9)^2 - 3 = 11(x^2 - 2). \quad \text{Ans. } x = \pm 5, \text{ or } \pm 2.$$

$$78. \sqrt{x + \sqrt{2x - 1}} - \sqrt{x - \sqrt{2x - 1}} =$$

$$\frac{3}{5} \sqrt{\frac{10x}{x + \sqrt{2x - 1}}} \quad \text{Ans. } x = \frac{4}{5}, \text{ or } \frac{6}{5}.$$

$$79. \frac{a - \sqrt{2ax - x^2}}{a + \sqrt{2ax - x^2}} = \frac{x}{a - x}. \quad \text{Ans. } x = a, \text{ or } \frac{a}{5}.$$

$$80. \frac{x^2 - 18}{x - 12} = \frac{4}{x^2}. \quad \text{Ans. } x = 4, -2, \text{ or } -1 \pm \sqrt{7}.$$

$$81. x + 4 + \left( \frac{x + 4}{x - 4} \right)^{\frac{1}{2}} = \frac{12}{x - 4}.$$

$$\text{Ans. } x = \pm 5, \text{ or } 4\sqrt{2}.$$

$$82. (x + 119)^{\frac{1}{2}} + (70 - x)^{\frac{1}{2}} = 9. \quad \text{Ans. } x = 6.$$

$$83. x^4 + x^3 - 4x^2 + x + 1 = 0.$$

$$\text{Ans. } x = 1, \text{ or } \frac{-3 \pm \sqrt{5}}{2}.$$

$$84. 2x^2 + \sqrt{x^2 + 9} = x^4 - 9.$$

$$\text{Ans. } x = \sqrt{\frac{3 \pm \sqrt{41}}{2}}, \text{ or } \sqrt{\frac{1 \pm \sqrt{37}}{2}}.$$

$$85. (a + x)\sqrt{a^2 + x^2} = 6(a - x)^2.$$

$$\text{Ans. } x = \frac{9 \pm 4\sqrt{2}}{7} \cdot a.$$

$$86. (x + 3)^2 - 2(x^2 + 8) = 2x(x + 1)^2.$$

$$\text{Ans. } x = 1, -3, \text{ or } -\frac{1}{2}.$$

$$87. \frac{x}{x^2 + 4x} + \frac{x}{x^2 - 3x} = 1\frac{1}{3}. \quad \text{Ans. } x = 4, \text{ or } -1\frac{2}{3}.$$

$$88. x^5 - 3x^4 - 9x^3 + 21x^2 - 10x + 24 = 0.$$

$$\text{Ans. } x = 2, 4, -3, \text{ or } \pm \sqrt{-1}.$$

$$89. \frac{x^2 + 1}{x} - \frac{1}{\sqrt{5}} \cdot \frac{x-1}{\sqrt{x}} = 4\frac{2}{5}.$$

$$\text{Ans. } x = 5, \frac{1}{5}, \text{ or } \frac{16 \mp 3\sqrt{29}}{10}.$$

$$90. 16(x^2 + 2)^{\frac{3}{2}} + \frac{9}{\sqrt{x^2 + 2}} = 32x^2 + 48. \quad x = \pm \frac{1}{2}.$$

$$91. x^{2m} - a = \frac{a}{x^m} + 1. \quad \text{Ans. } x = \sqrt[m]{\frac{1}{2} \pm \sqrt{a + \frac{1}{4}}}.$$

$$92. x^4 - 8x^3 + 10x^2 + 24x + 5 = 0.$$

$$\text{Ans. } x = 5, -1, \text{ or } 2 \pm \sqrt{5}.$$

$$93. \frac{1 + x^3}{(1 + x)^3} = \frac{1}{3}.$$

$$\text{Ans. } x = 2, \text{ or } \frac{1}{2}.$$

$$94. \frac{(x-a)^2}{\sqrt{x}} + 2(x-a) = \frac{a^2}{\sqrt{x}} + 1.$$

$$\text{Ans. } x = 2a + \frac{3}{2} \pm \sqrt{2a + \frac{5}{4}}.$$

$$95. \sqrt{x^2 - 1} + \frac{x\sqrt{x-1}}{\sqrt{x+1}} = \frac{\sqrt{(x+1)^3}}{\sqrt{x-1}}.$$

$$\text{Ans. } x = \frac{3 \pm \sqrt{7}}{2}.$$

$$96. (1 + x^3)(1 + x^2)(1 + x) = 30x^3.$$

$$\text{Ans. } x = \frac{3 \pm \sqrt{5}}{2}.$$

$$97. \frac{x^4}{2} + \frac{17x^3}{4} - 17x = 8. \quad \text{Ans. } x = -8, \text{ or } -\frac{1}{2}.$$

$$98. x^2(x^2 - 23) = 10x(x^2 - 24) + 649.$$

$$\text{Ans. } x = \frac{1}{2}(5 \pm 3\sqrt{29}).$$

$$99. \frac{1+x^3}{(1+x)^3} + \frac{1-x^3}{(1-x)^3} = a.$$

$$\text{Ans. } x = \left\{ \frac{a+4 \mp 2\sqrt{3(a+1)}}{a-2} \right\}^{\frac{1}{3}}.$$

$$100. x + 7\sqrt[3]{x} = 22. \quad x = 8, \text{ or } \{-1 \pm \sqrt{-10}\}^3.$$

$$101. \sqrt{x} - \frac{7}{\sqrt{x}-2} = \frac{8}{x}. \quad \text{Ans. } x = 1, \text{ or } 16.$$

$$102. x^4 \cdot \sqrt{x} + 2x^3 + 35x\sqrt{x} + 34 = \frac{72}{x\sqrt{x}}. \quad \text{Ans. } x = 1.$$

QUADRATICS WITH TWO OR MORE UNKNOWN QUANTITIES.

*Examples.*

$$1. \begin{cases} x + y = 10 \\ xy = 16 \end{cases} \dots\dots (1). \quad \text{Square (1), and multiply (2)} \\ \dots\dots (2). \quad \text{by 4.}$$

$$\begin{cases} x^2 + 2xy + y^2 = 100 \\ 4xy = 64 \end{cases} \quad \text{Subtract.}$$

$$\hline x^2 - 2xy + y^2 = 36 \quad \text{Extract the square root.}$$

$$(1) \quad \begin{cases} x - y = \pm 6 \\ x + y = 10 \end{cases} \quad \text{Add and subtract.}$$

$$\begin{array}{rcl} 2x & = & 16, \text{ or } 4; \quad \therefore x = 8, \text{ or } 2. \\ 2y & = & 4, \text{ or } 16; \quad \therefore y = 2, \text{ or } 8. \end{array}$$

This manner of proceeding may be adopted whenever the *sum* and *product* of the two unknown quantities are given.

$$2. \begin{cases} x - y = 3 \\ xy = 10 \end{cases} \dots\dots (1). \quad \text{Square (1), and multiply (2)} \\ \dots\dots (2). \quad \text{by 4.}$$

$$\begin{cases} x^2 - 2xy + y^2 = 9 \\ 4xy = 40 \end{cases} \quad \text{Add.}$$

$$\hline x^2 + 2xy + y^2 = 49 \quad \text{Extract the square root.}$$

$$(1) \quad \begin{cases} x + y = \pm 7 \\ x - y = 3 \end{cases} \quad \text{Add and subtract.}$$

$$\begin{array}{rcl} 2x & = & 10, \text{ or } -4; \quad \therefore x = 5, \text{ or } -2. \\ 2y & = & 4, \text{ or } -10; \quad \therefore y = 2, \text{ or } -5. \end{array}$$





$$6. \begin{cases} x^4 + y^4 = 337 \\ x + y = 7 \end{cases} \dots\dots (1). \quad \text{Raise (2) to the 4th power.}$$

$$(1) \quad \begin{cases} x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 = 2401 \\ x^4 + y^4 = 337 \end{cases} \quad \text{Subtract.}$$

$$4x^3y + 6x^2y^2 + 4xy^3 = 2064 \quad \text{Divide by } 4xy.$$

$$x^2 + \frac{3}{2}xy + y^2 = \frac{516}{xy} \quad \text{Square (2).}$$

$$x^2 + 2xy + y^2 = 49 \quad \text{Subtract.}$$

$$\therefore \quad \frac{1}{2}xy = 49 - \frac{516}{xy}.$$

Multiply by  $2xy$ , and transpose.

$$x^2y^2 - 98xy = -1032. \quad \text{Complete the square.}$$

$$x^2y^2 - 98xy + 49^2 = 2401 - 1032 = 1369,$$

$$xy - 49 = \pm 37;$$

$$\therefore \quad xy = 49 \pm 37 = 86, \text{ or } 12.$$

Taking  $xy = 12$ , and  $x + y = 7$ , and proceeding as in *Ex. 1*, we have  $x = 4$ , or  $3$ ;  $y = 3$ , or  $4$ .

$$7. \begin{cases} x^2 + y^2 + x + y = 922 \\ \sqrt{xy} = 20 \end{cases} \dots\dots (1).$$

$$\sqrt{xy} = 20 \dots\dots (2).$$

$$xy = 400$$

$$2xy = 800. \quad \text{Adding this to (1),}$$

$$x^2 + 2xy + y^2 + x + y = 1722. \quad \text{Complete the square.}$$

$$\left( (x+y)^2 + (x+y) + \frac{1}{2} \right)^2 = 1722 + \frac{1}{4} = \frac{6889}{4},$$

$$x + y + \frac{1}{2} = \pm \frac{83}{2};$$

$$\therefore x + y = \frac{\pm 83 - 1}{2} = \frac{82}{2}, \text{ or } -\frac{84}{2} = 41, \text{ or } -42;$$

and  $\therefore xy = 400$ , by proceeding as in *Ex. 1*, we have  $x = 25$ ,  $y = 16$ .

$$8. \left. \begin{array}{l} x^2 + 3y + x = 75 - 2xy \\ y^2 - y + x = 24 \end{array} \right\} \dots\dots (1).$$

$$\left. \begin{array}{l} y^2 - y + x = 24 \end{array} \right\} \dots\dots (2).$$

$$\left. \begin{array}{l} x^2 + 2xy \quad + \quad x + 3y = 75 \\ \quad \quad y^2 + \quad x - \quad y = 24 \end{array} \right\} \dots\dots (3), \quad \text{Add.}$$

$$\hline x^2 + 2xy + y^2 + 2x + 2y = 99,$$

$$(x + y)^2 + 2(x + y) = 99, \text{ Complete the square.}$$

$$(x + y)^2 + 2(x + y) + 1 = 100,$$

$$x + y + 1 = \pm 10,$$

$$x + y = \pm 10 - 1 = 9, \text{ or } -11;$$

$$\therefore x = 9 - y. \text{ Substitute in (3).}$$

$$y^2 - y + 9 - y = 24, \quad y^2 - 2y = 15,$$

$$y^2 - 2y + 1 = 16, \quad y - 1 = \pm 4;$$

$$\therefore y = 1 \pm 4 = 5, \text{ or } -3; \text{ and } x = 4, \text{ or } 12.$$

$$9. \left. \begin{array}{l} x^3 - y^2 = xy \\ x^2 + y^2 = x^3 - y^3 \end{array} \right\} \dots\dots (1). \text{ Multiply (1) by } x + y.$$

$$\left. \begin{array}{l} x^2 + y^2 = x^3 - y^3 \end{array} \right\} \dots\dots (2).$$

$$\left. \begin{array}{l} x^3 + x^2y - xy^2 - y^3 = xy(x + y) \\ x^3 \quad \quad \quad - y^3 = x^2 + y^2 \end{array} \right\} \text{ Subtract.}$$

$$x^2y - xy^2 = x^2y + xy^2 - x^2 - y^2, \text{ Transpose.}$$

$$\left. \begin{array}{l} x^2 + y^2 = 2xy^2 \\ x^2 - y^2 = xy \end{array} \right\} \text{ Add.}$$

$$2x^2 = 2xy^2 + xy. \text{ Divide by } 2x.$$

$$\left. \begin{array}{l} x = y^2 + \frac{y}{2} \\ x^2 = y^4 + y^3 + \frac{y^2}{4} \end{array} \right\} \text{ Substitute these values of } x \text{ and } x^2 \text{ in (1).}$$

$$y^4 + y^3 + \frac{y^2}{4} - y^2 = y^3 + \frac{y^2}{2},$$

$$y^4 = y^2 + \frac{y^2}{2} - \frac{y^2}{4},$$

$$y^4 = \frac{5y^2}{4}, \quad y^2 = \frac{5}{4}, \quad \therefore y = \frac{\sqrt{5}}{2},$$

$$\text{and } x = \frac{5}{4} + \frac{\sqrt{5}}{4} = \frac{\sqrt{5}(\sqrt{5} + 1)}{4}.$$

$$\begin{array}{rcl}
 10. \quad \left. \begin{array}{l} 5 + \sqrt{xy} = 4\sqrt{y+2} \quad \dots\dots (1). \\ 8 + x = \sqrt{128 + 64y} \quad \dots\dots (2). \end{array} \right\} & & \\
 (2) \quad \left. \begin{array}{l} 8 + x = \sqrt{64(2+y)}, \\ 8 + x = 8\sqrt{y+2} \end{array} \right\} & \text{Subtract.} & \\
 (1) \quad 10 + 2x^{\frac{1}{2}}y^{\frac{1}{2}} = 8\sqrt{y+2} & &
 \end{array}$$

$$2 + 2x^{\frac{1}{2}}y^{\frac{1}{2}} - x = 0,$$

$$\therefore 2x^{\frac{1}{2}}y^{\frac{1}{2}} = x - 2, \quad \therefore y^{\frac{1}{2}} = \frac{x-2}{2x^{\frac{1}{2}}},$$

$$\therefore y = \frac{(x-2)^2}{4x}, \quad \therefore 64y = \frac{16(x-2)^2}{x} \quad \text{Square (2).}$$

$$64 + 16x + x^2 = 64y + 128,$$

$$\therefore x^2 + 16x - 64 = \frac{16(x-2)^2}{x},$$

$$x^3 + 16x^2 - 64x = 16x^2 - 64x + 64,$$

$$x^3 = 64, \quad \therefore x = 4,$$

$$y = \frac{(4-2)^2}{16} = \frac{2^2}{16} = \frac{4}{16} = \frac{1}{4}.$$

$$11. \quad \left. \begin{array}{l} x^4 = 3x + 2y \\ y^4 = 2x + 3y \end{array} \right\} \quad \text{Add and subtract.}$$

$$x^4 + y^4 = 5(x+y), \quad (1), \quad x^4 - y^4 = x - y, \quad (2),$$

$$(2) \quad (x^2 + y^2)(x^2 - y^2) = x - y, \quad \text{Divide by } x - y.$$

$$(x^2 + y^2)(x + y) = 1 \dots\dots (3), \quad \therefore x + y = \frac{1}{x^2 + y^2},$$

$$(1) \quad x^4 + y^4 = \frac{5}{x^2 + y^2}, \quad \therefore (x^4 + y^4)(x^2 + y^2) = 5 \dots (4),$$

$$\frac{(x^2 + y^2)^2 (x + y)^2}{(x^4 + y^4)(x^2 + y^2)} = \frac{1}{5}, \quad \text{by squaring (3) and dividing by (4),}$$

$$\therefore \frac{(x^2 + y^2)(x + y)^2}{x^4 + y^4} = \frac{1}{5},$$

$$\text{or } \frac{x^4 + 2x^2y^2 + y^4 + 2x^3y + 2xy^3}{x^4 + y^4} = \frac{1}{5}.$$

Clearing this equation, and dividing by  $4x^2y^2$ , we have

$$\frac{x^2}{y^2} + \frac{5}{2} + \frac{y^2}{x^2} + \frac{5}{2} \cdot \frac{x}{y} + \frac{5}{2} \cdot \frac{y}{x} = 0, \quad \text{Subtract } \frac{1}{2}.$$

$$\frac{x^2}{y^2} + 2 + \frac{y^2}{x^2} + \frac{5}{2} \left( \frac{x}{y} + \frac{y}{x} \right) = -\frac{1}{2},$$

$$\text{or } \left( \frac{x}{y} + \frac{y}{x} \right)^2 + \frac{5}{2} \left( \frac{x}{y} + \frac{y}{x} \right) = -\frac{1}{2}.$$

Completing the square, and extracting the root, we have

$$\frac{x}{y} + \frac{y}{x} = \frac{-5 \pm \sqrt{17}}{4}. \quad \text{Multiply by } \frac{x}{y}, \text{ and transpose.}$$

$$\frac{x^3}{y^2} + \frac{5 \mp \sqrt{17}}{4} \cdot \frac{x}{y} = -1. \quad \text{Complete the square, and extract the root.}$$

$$\frac{x}{y} = -\frac{5 \mp \sqrt{17} \mp \sqrt{-22 \mp 10\sqrt{17}}}{8}.$$

$$\begin{aligned} (2) \quad (x^2 + y^2)(x + y)(x - y) - (x - y) &= 0, \\ \{(x^2 + y^2)(x + y) - 1\}(x - y) &= 0, \\ \therefore x - y &= 0, \quad \therefore x = y. \end{aligned}$$

It is evident from the last part of this solution, that any factor common to all the terms of an equation is equal to 0.

$$\text{Again, } x^4 = 3x + 2x = 5x, \quad \therefore x^3 = 5, \quad \therefore x = \sqrt[3]{5}.$$

When the *sum of the indices of the unknown quantities* is the same in every term, the equations are called *homogeneous*, and they may be solved, when no simpler solution presents itself, by assuming one of them to be equal to some unknown multiple of the other, as in the following example,

$$\left. \begin{aligned} 12. \quad x^2 + y^2 &= 34 \\ x^2 - xy &= 10 \end{aligned} \right\} \quad \begin{aligned} &\text{Assume } x = vy, \text{ then } x^2 = v^2y^2, \\ &\text{and } xy = vy^2, \end{aligned}$$

$$\therefore v^2y^2 + y^2 = 34, \quad (v^2 + 1)y^2 = 34, \quad \therefore y^2 = \frac{34}{v^2 + 1},$$

$$v^2y^2 - vy^2 = 10, \quad (v^2 - v)y^2 = 10, \quad \therefore y^2 = \frac{10}{v^2 - v}.$$

Hence,  $\frac{34}{v^2 + 1} = \frac{10}{v^2 - v}$ , or  $\frac{17}{v^2 + 1} = \frac{5}{v^2 - v}$ ,

$$17v^2 - 17v = 5v^2 + 5, \quad 12v^2 - 17v = 5,$$

$\therefore v^2 - \frac{17}{12}v = \frac{5}{12}$ , and solving this quadratic, we have  $v = \frac{5}{3}$ ,

$$\therefore y^2 = \frac{10}{v^2 - v} = \frac{10}{\frac{25}{9} - \frac{5}{3}} = \frac{90}{25 - 15} = \frac{90}{10} = 9,$$

$\therefore y = 3$ , and  $x = vy = \frac{5}{3} \times 3 = 5$ .

13.  $\left. \begin{aligned} x^2 + xy + y^2 &= 37 \\ x^2 + xz + z^2 &= 28 \\ y^2 + yz + z^2 &= 19 \end{aligned} \right\} \begin{aligned} &\dots (A). \\ &\dots (B). \\ &\dots (C). \end{aligned}$  The method adopted in the last example would apply to this; we will, however, give another solution.

$\left. \begin{aligned} A - B, \quad y^2 - z^2 + (y - z)x &= 9 \\ B - C, \quad x^2 - y^2 + (x - y)z &= 9 \end{aligned} \right\}, \text{ or}$

$$\left. \begin{aligned} y + z + x &= \frac{9}{y - z} \\ x + y + z &= \frac{9}{x - y} \end{aligned} \right\} \dots\dots (D).$$

$\therefore \frac{9}{y - z} = \frac{9}{x - y}, \quad \text{or } y - z = x - y,$

$\therefore x + z = 2y$ , Substitute in (D).

$3y = \frac{9}{y - z}, \quad y = \frac{3}{y - z}, \quad y^2 - yz = 3,$

$yz = y^2 - 3, \quad \therefore z = \frac{y^2 - 3}{y} \left\{ \begin{aligned} &\text{Substitute in (C).} \\ &z^2 = \frac{y^4 - 6y^2 + 9}{y^2} \end{aligned} \right.$

$y^2 + y^2 - 3 + \frac{y^4 - 6y^2 + 9}{y^2} = 19,$

$$2y^2 + \frac{y^4 - 6y^2 + 9}{y^2} = 22,$$

$$2y^4 + y^4 - 6y^2 + 9 = 22y^2,$$

$$3y^4 - 28y^2 = -9,$$

$$y^4 - \frac{28}{3}y^2 = -3, \text{ and, solving this quadratic, we have}$$

$$y = \pm 3, \quad \therefore z = y - \frac{3}{y} = \pm 3 - \frac{3}{\pm 3} = \pm 2,$$

$$\text{and } x = 2y - z = \pm 6 \mp 2 = \pm 4.$$

### Examples.

1.  $\begin{cases} x^2 + y^2 = 20. \\ x^2 - y^2 = 12. \end{cases}$  Ans.  $x = 4.$   
 $y = 2.$
2.  $\begin{cases} x + y = 6. \\ x^2 + y^2 = 26. \end{cases}$  Ans.  $x = 5.$   
 $y = 1.$
3.  $\begin{cases} x^2 + y^2 = 10. \\ x - y = 2. \end{cases}$  Ans.  $x = 3.$   
 $y = 1.$
4.  $\begin{cases} x^2 + y^2 = 25. \\ x + y = 1. \end{cases}$  Ans.  $x = 4, \text{ or } -3.$   
 $y = -3, \text{ or } 4.$
5.  $\begin{cases} x^2 - y^2 = 16. \\ x + y = 8. \end{cases}$  Ans.  $x = 5.$   
 $y = 3.$
6.  $\begin{cases} x - y = 1. \\ x^2 - y^2 = 19. \end{cases}$  Ans.  $x = 3, \text{ or } -2.$   
 $y = 2, \text{ or } -3.$
7.  $\begin{cases} x^3 + y^3 = 189. \\ x^2y + xy^2 = 180. \end{cases}$  Ans.  $x = 5, \text{ or } 4.$   
 $y = 4, \text{ or } 5.$
8.  $\begin{cases} 10x + y = 3xy. \\ y - x = 2. \end{cases}$  Ans.  $x = 2, \text{ or } -\frac{1}{3}.$   
 $y = 4, \text{ or } -\frac{5}{3}.$
9.  $\begin{cases} x^2 + y^2 + x + y = 18. \\ 2xy = 12. \end{cases}$  Ans.  $x = 3, \text{ or } 2, \text{ or } -3 \pm \sqrt{3}.$   
 $y = 2, \text{ or } 3, \text{ or } -3 \mp \sqrt{3}.$
10.  $\begin{cases} x^2y^4 + y^2 = 10. \\ xy^3 + y = 4. \end{cases}$  Ans.  $x = 3.$   
 $y = 1.$

$$11. \left. \begin{aligned} \frac{x+y}{x-y} &= a^2 \\ x^2 - y^2 &= b^2 \end{aligned} \right\} \quad \begin{aligned} \text{Ans. } x &= \frac{b}{2}(a^2 + 1). \\ x &= \frac{b}{2}(a^2 - 1). \end{aligned}$$

$$12. 9x^2 = 4y^2, \quad 3xy + 2x + y = 485. \\ \text{Ans. } x = 10, \text{ or } -10\frac{7}{9}. \quad y = 15, \text{ or } -16\frac{1}{6}.$$

$$13. \left. \begin{aligned} x^2 + y^2 - x - y &= 78. \\ xy + x + y &= 39. \end{aligned} \right\} \quad \begin{aligned} \text{Ans. } x &= \frac{-13 \pm \sqrt{-39}}{2} \\ y &= \frac{-13 \mp \sqrt{+39}}{2}. \end{aligned}$$

$$14. \left. \begin{aligned} \frac{1}{y} - \frac{1}{x} &= \frac{1}{4}. \\ x^2y - xy^2 &= 16. \end{aligned} \right\} \quad \begin{aligned} \text{Ans. } x &= 4, \text{ or } -2. \\ y &= 2, \text{ or } -4. \end{aligned}$$

$$15. \left. \begin{aligned} x^2 + xy &= a^2 + ab. \\ y^2 + yx &= b^2 + ab. \end{aligned} \right\} \quad \begin{aligned} \text{Ans. } x &= a. \\ y &= b. \end{aligned}$$

$$16. \left. \begin{aligned} 12xy &= 5x + 12y. \\ y^2 - x^2 &= 1. \end{aligned} \right\} \quad \begin{aligned} \text{Ans. } x &= 1\frac{1}{3}. \\ y &= 1\frac{2}{3}. \end{aligned}$$

$$17. \left. \begin{aligned} 2y + 3x &= 8. \\ 3y^2 + 2x^2 &= 11. \end{aligned} \right\} \quad \begin{aligned} \text{Ans. } x &= 2, 2\frac{4}{5}. \\ y &= 1, \frac{2}{5}. \end{aligned}$$

$$18. \left. \begin{aligned} y - x &= 2. \\ 3xy &= 10x + y. \end{aligned} \right\} \quad \begin{aligned} \text{Ans. } x &= 2, -\frac{1}{3}. \\ y &= 4, 1\frac{2}{3}. \end{aligned}$$

$$19. \left. \begin{aligned} x + y + \sqrt{x+y} &= 12. \\ x^3 + y^3 &= 189. \end{aligned} \right\} \quad \begin{aligned} \text{Ans. } x &= 5, \text{ or } 4. \\ y &= 4, \text{ or } 5. \end{aligned}$$

$$20. \left. \begin{aligned} 4xy &= 96 - x^2y^2. \\ x + y &= 6. \end{aligned} \right\} \quad \begin{aligned} \text{Ans. } x &= 4, \text{ or } 2, \text{ or } 3 \pm \sqrt{21}. \\ y &= 2, \text{ or } 4, \text{ or } 3 \mp \sqrt{21}. \end{aligned}$$

$$21. \left. \begin{aligned} \frac{x^2 + y^2}{xy} &= \frac{13}{6}. \\ x^3y + xy^3 &= x^2y^2 + 42. \end{aligned} \right\} \quad \begin{aligned} \text{Ans. } x &= 3. \\ y &= 2. \end{aligned}$$

$$22. \left. \begin{aligned} x^2 + y^2 + 4x - 6y &= 13. \\ xy - 3x + 2y &= 11. \end{aligned} \right\} \quad \begin{aligned} \text{Ans. } x &= 3, -1, -3, -7. \\ y &= 4, 8, -2, 2. \end{aligned}$$

$$23. \left. \begin{aligned} (x^2 + y^2)x^2y^2 &= 3600. \\ x^2y + xy^2 &= 84. \end{aligned} \right\} \quad \begin{aligned} \text{Ans. } x &= 3, \text{ or } 4. \\ y &= 4, \text{ or } 3. \end{aligned}$$



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$$24. \begin{cases} x^4 + y^4 = 2657. \\ x + y = 11. \end{cases} \quad \text{Ans. } x = 7, \text{ or } 4. \\ y = 4, \text{ or } 7.$$

$$25. \begin{cases} 2x + 2y - x^2 - y^2 + 2 = 0. \\ xy = 3. \end{cases} \quad \text{Ans. } x = 3, \text{ or } 1, \text{ or } -(1 \pm \sqrt{-2}). \\ y = 1, \text{ or } 3, \text{ or } -(1 \mp \sqrt{-2}).$$

$$26. \begin{cases} x^2 + y^2 = 61. \\ x^2 - xy = 6. \end{cases} \quad \text{Ans. } x = 6. \\ y = 5.$$

$$27. \begin{cases} x + y = 10. \\ x^5 + y^5 = 17050. \end{cases} \quad \text{Ans. } x = 7, \text{ or } 3. \\ y = 3, \text{ or } 7.$$

$$28. \begin{cases} x^3 + xy + y^3 = 21. \\ x - x^{\frac{1}{2}}y^{\frac{1}{2}} + y = 3. \end{cases} \quad \text{Ans. } x = 1. \\ y = 4.$$

$$29. \begin{cases} x^2y^2 + 12xy = 9x^2 + 4y^2. \\ x^2 + 4x + y^2 = 6y + 24. \end{cases} \quad \text{Ans. } x = 4, \text{ or } -3. \\ y = 2, \text{ or } 9.$$

$$30. \begin{cases} x - y = 2. \\ x^4 + y^4 = 272. \end{cases} \quad \begin{cases} x = 4, -2, \text{ or } \sqrt{-15} + 1. \\ y = 2, -4, \text{ or } \sqrt{-15} - 1. \end{cases}$$

$$31. \begin{cases} x^3 + 2xy + y + 3x = 73. \\ y^3 + x + 3y = 44. \end{cases} \quad \text{Ans. } x = 4, \text{ or } 16. \\ y = 5, \text{ or } -7.$$

$$32. \begin{cases} (x^2 + y^2) \cdot (x + y) = 120. \\ (x - y) \cdot (x^2 - y^2) = 24. \end{cases} \quad \text{Ans. } x = 4. \\ y = 2.$$

$$33. \begin{cases} xy = 6. \\ 3x^2 - 7y^2 + 1 = 0. \end{cases} \quad \text{Ans. } x = \pm 3. \\ y = \pm 2.$$

$$34. \begin{cases} 2x^{-\frac{1}{2}}y^{-\frac{1}{2}} = 1 + x^{\frac{1}{2}}y^{\frac{1}{2}} \\ x - y = \sqrt{a - x + y} - \sqrt{a + x - y} \end{cases} \quad x = y = 1.$$

$$35. \begin{cases} x - y = 3, & x^3 + y^3 = 19(x + y). \end{cases} \quad \text{Ans. } x = 5, -2, \quad y = 2, -5$$

$$36. \begin{cases} x - 2\sqrt{xy} + y - \sqrt{x} + \sqrt{y} = 0. \\ \sqrt{x} + \sqrt{y} = 5. \end{cases} \quad \begin{cases} x = 9, \text{ or } \frac{25}{4}. \\ y = 4, \text{ or } \frac{25}{4}. \end{cases}$$

$$37. \left. \begin{aligned} \frac{x^2}{y^2} + \frac{4x}{y} &= 9\frac{1}{4}. \\ x - y &= 2. \end{aligned} \right\} \quad \begin{aligned} \text{Ans. } x &= 5. \\ y &= 3. \end{aligned}$$

$$38. \left. \begin{aligned} x^4 - 2x^2y + y^2 &= 49. \\ x^4 - 2x^2y^2 + y^4 - x^2 + y^2 &= 20. \end{aligned} \right\} \quad \begin{aligned} \text{Ans. } x &= \pm 3, \text{ or } \pm \sqrt{6}. \\ y &= 2, \text{ or } -1. \end{aligned}$$

$$39. \left. \begin{aligned} x^2 + y^2 &= a \\ xy &= b \end{aligned} \right\} \quad \begin{aligned} \text{Ans. } x &= \sqrt{\frac{1}{2}(\sqrt{a^2 - 4b^2} + a)}. \\ y &= \sqrt{\frac{1}{2}(\sqrt{a^2 - 4b^2} - a)}. \end{aligned}$$

$$40. \left. \begin{aligned} \frac{x^3}{y^2} + \frac{y^2}{x^2} - \frac{3}{2}\left(\frac{x}{y} + \frac{y}{x}\right) &= 2\frac{3}{5}. \\ 4x - 5y &= 10. \end{aligned} \right\} \quad \begin{aligned} \text{Ans. } x &= 5. \\ y &= 2. \end{aligned}$$

$$41. \begin{aligned} x^2 - xy &= 48y, & xy - y^2 &= 3x. \\ \text{Ans. } x &= 16, -9\frac{1}{2}, & y &= 4, \text{ or } 2\frac{2}{3}. \end{aligned}$$

$$42. \left. \begin{aligned} x^2 + y^2 - 15(x + y) &= -70. \\ 3xy + 31(x + y) &= 210. \end{aligned} \right\} \quad \begin{aligned} \text{Ans. } x &= 2. \\ y &= 4. \end{aligned}$$

$$43. \left. \begin{aligned} \sqrt{y} : \sqrt{x} :: \sqrt{x} + 3 : \sqrt{x} + 1. \\ \sqrt{xy} + 2\sqrt{y} &= 3x + 3\sqrt{x}. \end{aligned} \right\} \quad \begin{aligned} x &= 1. \\ y &= 4. \end{aligned}$$

$$44. \left. \begin{aligned} \frac{x}{y} - \frac{y}{x} &= \frac{11}{30}. \\ x^2 + xy &= 66. \end{aligned} \right\} \quad \begin{aligned} \text{Ans. } x &= 6. \\ y &= 5. \end{aligned}$$

$$45. \left. \begin{aligned} 3x^2 &= 2xy + 24. \\ y^2 &= xy - 3. \end{aligned} \right\} \quad \begin{aligned} \text{Ans. } x &= 4. \\ y &= 3. \end{aligned}$$

$$66. \left. \begin{aligned} (x+y)^3 &= 64(x-y) \\ (x^3+y^3)(x+y) &= 76. \end{aligned} \right\} \quad \text{Ans. } x = \frac{5}{2}, \\ y = \frac{3}{2}.$$

$$67. \left. \begin{aligned} x^4+y^4 &= 1+2xy+3x^2y^2 \\ x^3+y^3 &= 2y^2x+2y^2+x+1. \end{aligned} \right\} \quad \text{Ans. } x = 2, \\ y = 1.$$

$$68. \left. \begin{aligned} (x^2+xy+y^2)(x^2-xy+y^2) &= 481 \\ (x^3+y^3)^2 - (x^2+y^2)xy &= 325. \end{aligned} \right\} \quad \text{Ans. } x = 3, \\ y = 4.$$

$$69. \left. \begin{aligned} x - 2\sqrt{2ay-y^2} &= 5\sqrt{ay} \\ x^2 - 8\sqrt{ay} \cdot x &= 2ay - 9y^2. \end{aligned} \right\} \quad \text{Ans. } x = 7a, \\ y = a.$$

$$70. \left. \begin{aligned} x^6+y^6 &= \frac{1}{\sqrt{2}} \\ x^4+y^4 &= 1. \end{aligned} \right\} \quad x = y = \frac{1}{\sqrt[4]{2}}.$$

$$71. \left. \begin{aligned} x^7-y^7 &= 127 \\ x-y &= 1. \end{aligned} \right\} \quad \text{Ans. } x = 2, \\ y = 1.$$

$$72. \left. \begin{aligned} x+y+z &= 3a \\ xy+xz+yz &= 3a^2 \\ xyz &= a^3. \end{aligned} \right\} \quad \text{Ans. } x = a, \\ y = a, \\ z = a.$$

$$73. \left. \begin{aligned} xy &= x+y \\ xz &= 2(x+2) \\ yz &= 3(y+z). \end{aligned} \right\} \quad \text{Ans. } x = 1\frac{5}{7}, \\ y = 2\frac{2}{7}, \\ z = -12.$$

$$74. \left. \begin{aligned} x^2+y^2+z^2 &= 14 \\ x^2+y &= 3 \\ z^2+y &= 11. \end{aligned} \right\} \quad \text{Ans. } x = \pm 1 \\ y = 2, \\ z = \pm 3.$$

$$75. \left. \begin{aligned} xy+z &= 5 \\ xyz &= 4 \\ 2(x^2-y) &= (y^2-x)^2. \end{aligned} \right\} \quad \text{Ans. } x = 2, -2\sqrt[3]{4} \pm 4, \\ y = 2, -\sqrt[3]{2} \pm 1, \\ z = 1, 4.$$

$$76. \left. \begin{aligned} x^2+xy+y^2 &= 18 \\ y^2+yz+z^2 &= 49 \\ x^2+xz+z^2 &= 31. \end{aligned} \right\} \quad \text{Ans. } x = \pm 1, \\ y = \pm 3, \\ z = \pm 5.$$

$$77. \left. \begin{aligned} x^{-2}y^{-1}z &= 1\frac{1}{2} \\ x^{-1}ys^2 &= 18 \\ xy^3z^3 &= 108. \end{aligned} \right\} \quad \text{Ans. } x = \pm 1, \\ y = \pm 2, \\ z = \pm 3.$$

$$\begin{aligned}
 78. \left. \begin{aligned} (\sqrt{x} + \sqrt{y})^2 &= y^2 - 4xz. \\ \sqrt{x} + \sqrt{y} &= \frac{x + y + z}{2\sqrt{x}}. \\ 6\sqrt{y} &= x + y^{\frac{1}{2}} + z. \end{aligned} \right\} \begin{aligned} \text{Ans. } x &= 4. \\ y &= 16. \\ z &= 12. \end{aligned}
 \end{aligned}$$

*Problems.*

1. A sets out from London to York, B from York to London; A arrives in York 9 hours, and B in London 16 hours, after they met. In what time did each perform the journey?

Let  $x$  = number of hours each was on the road before they met,

$$\begin{aligned}
 \text{then } x + 9 &= \text{A's whole time,} \\
 x + 16 &= \text{B's,}
 \end{aligned}$$

hours. hours. journey.

$$x + 9 : x :: 1 : \frac{x}{x + 9} = \text{distance A goes before they met,}$$

$$x + 16 : x :: 1 : \frac{x}{x + 16} = \text{,, B ,,}$$

$$\text{Hence, } \frac{x}{x + 9} + \frac{x}{x + 16} = 1,$$

$$x^2 + 16x + x^2 + 9x = x^2 + 25x + 144,$$

$$\therefore x^2 = 144, x = 12,$$

$$\therefore x + 9 = 21 = \text{A's time,}$$

$$x + 16 = 28 = \text{B's ,, .}$$

Or thus: let  $x$  be A's whole time,

$y$  ... B's .....

$$x : 9 :: 1 : \frac{9}{x}, \text{ A's distance after they met,}$$

$$x : 16 :: 1 : \frac{16}{y}, \text{ B's, .....$$

$$\text{Hence, } \frac{9}{x} + \frac{16}{y} = 1, \therefore 9y + 16x = xy.$$

But  $x = y - 7$ ,  $\therefore$  by substitution,

$$\begin{aligned}
 9y + 16y - 112 &= y^2 - 7y, \\
 y^2 - 32y &= -112,
 \end{aligned}$$

$$\therefore y = 28 \text{ hours} = \text{B's time, } x = y - 7 = 21 \text{ hours} = \text{A's time.}$$

2. A and B set out at the same time, A from C to go to D, and B from D to go to C; they meet on the road, when it appears that A has travelled 30 miles more than B, and that, at the rate he is travelling, he will reach D in 4 days, and that B will arrive at C in 9 days. Find the distance of C from D.

Let  $x$  = A's distance before they met,

$y$  = B's .....

$$\therefore x = y + 30.$$

Now  $y$  = miles A goes in 4 days,

$$y : x :: 4 : \frac{4x}{y} = \text{N}^\circ. \text{ of days A is travelling before they meet.}$$

Again,  $x$  = miles B goes in 9 days,

$$\therefore x : y :: 9 : \frac{9y}{x} = \text{N}^\circ. \text{ of days B is travelling before they meet.}$$

$$\text{Hence, } \frac{4x}{y} = \frac{9y}{x};$$

$$\therefore 4x^2 = 9y^2, \quad \therefore 2x = 3y, \quad \therefore x = \frac{3y}{2}.$$

$$\text{Hence, } \frac{3y}{2} = y + 30, \quad \therefore y = 60, \text{ and } x = 90,$$

$$\therefore x + y = 150 \text{ miles} = \text{the distance.} \quad \text{Ans.}$$

3. What are those two numbers whose difference is 2, and whose product multiplied by their sum is 12?

Let  $x$  = the greater, and  $y$  = the less,

$$\text{then } x - y = 2, \quad \text{or } y = x - 2,$$

$$\text{and } xy(x + y) = 12, \quad \text{or } x^2y + xy^2 = 12,$$

$$x^2(x - 2) + x(x - 2)^2 = 12,$$

$$x^3 - 2x^2 + x^3 - 4x^2 + 4x = 12,$$

$$2x^3 - 6x^2 + 4x - 12 = 0,$$

$$2x^2(x - 3) + 4(x - 3) = 0,$$

$$\therefore x = 3, \quad y = 1.$$

4. The sum of two numbers multiplied by the sum of their cubes is 112, and the cube of their sum is to their difference as 32 to 1; find the numbers.

Let  $x$  = the greater, and  $y$  = the less,

$$\text{then } (x + y)(x^3 + y^3) = 112,$$

$$\text{and } (x + y)^3 : x - y :: 32 : 1.$$

$$(x + y)^4 : x^2 - y^2 :: 32 : 1, \text{ or}$$

$$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 = 32(x^2 - y^2),$$

$$4x^4 + 4x^3y \qquad + 4xy^3 + 4y^4 = 448$$

$$3x^4 \qquad - 6x^2y^2 \qquad + 3y^4 = 448 - 32(x^2 - y^2)$$

$$x^4 - 2x^2y^2 + y^4 + \frac{32}{3}(x^2 - y^2) = \frac{448}{3},$$

$$(x^2 - y^2)^2 + \frac{32}{3}(x^2 - y^2) + \frac{16}{3} = \frac{256}{9} + \frac{1344}{9} = \frac{1600}{9},$$

$$x^2 - y^2 + \frac{16}{3} = \frac{40}{3}, \quad x^2 - y^2 = \frac{24}{3} = 8,$$

$$\therefore (x + y)^4 = 32(x^2 - y^2) = 256,$$

$$x + y = 4, \quad x - y = 2,$$

$$\therefore x = 3, \quad y = 1.$$

*Problems producing Equations.*

1. Required two numbers whose difference is 8, and product 128. Ans.  $\pm 8$ , and  $\pm 16$ .

2. There are two numbers whose sum is 40, and the sum of their squares is 818; find them. Ans. 23 and 17.

3. What magnitude is that which exceeds its reciprocal by 1? Ans.  $\frac{1}{2}(1 \pm \sqrt{5})$ .

4. A person bought as many sheep as cost him 60*l.*, and after reserving 15 out of the number, sold the remainder for 54*l.*, and gained 2*s.* a head by them; how many did he buy? Ans. 75.

5. A, B, and C together perform a piece of work in a certain time; A alone could have done it in 6 hours more, B alone in 15 hours more, and C alone in twice the time; how long did it occupy them? Ans. 3 hours.

6. In a circular grass plot, a walk, A B, cuts another, C D,



into two equal parts in the point E;  $AE = 25$  feet,  $EB =$  of  $CD = 16$  feet; required the length of  $CD$ .   Ans. 40 feet

7. A grazier bought a certain number of oxen for 240*l*. and after losing 3, sold the remainder at 8*l*. a head more than they cost him, thus gaining 59*l*. by his bargain; what number did he buy?   Ans. 16

8. The reckoning of a party at a tavern was 3*l*. 12*s*., but in consequence of two of them having no money, each of the rest paid 6*d*. more than he otherwise should have done required their number.   Ans. 18

9. A person at play won twice as much as he began with and then lost 16 shillings; after this he lost four-fifths of what remained, and then won as much as he began with, and counting his money found he had 80 shillings; what sum did he begin with?   Ans. 52*s*

10. There is a field in the form of a rectangular parallelogram, whose length exceeds the breadth by 10 yards, and it contains 3000 square yards; required the length and breadth.   Ans. 60 and 50 yards

11. Two partners, A and B, gained 18*l*. by trade; A's money was in trade 12 months, and he received for his principal and gain 26*l*; also B's money, which was 30*l*., was in trade 16 months; what money did A put into trade?   Ans. 20*l*

12. There are two square buildings that are paved with stones a foot square each, the side of one building exceeds that of the other by 12 feet, and both their pavements together contain 2120 stones; what are the lengths of them separately?   Ans. 26 and 38 feet respectively.

13. A person by selling a horse for 56*l*., gains as much per cent. as the horse cost him; what was its original price?   Ans. 40*l*

14. The difference of two numbers is 6, and the sum of their squares multiplied by their product is 4640; find them.   Ans. 10 and 4.

15. A and B start at the same time to travel 150 miles, A travels 3 miles an hour faster than B, and finishes his journey  $8\frac{1}{2}$  hours before him; what is the rate of each?   Ans. 9 and 6 miles per hour.

16. A man working for 10 hours, assisted by a boy who works for 6 hours, does a certain piece of work; if the man had worked for 6 hours and the boy for 10, only two-thirds

of the work would have been done; how long will it take the man and the boy to do the work, supposing the man to work 5 hours longer than the boy?

Ans. man,  $10\frac{1}{10}$  hours, boy,  $5\frac{1}{10}$  hours.

17. A person rows 20 miles down a river and back again in 10 hours, and he finds that he can row two miles against the stream in the same time that he can row three miles with it; required the rate of the stream, and the times of his going and returning.

Ans.  $\frac{1}{2}$  of a mile per hour, and 4 and 6 hours.

18. A body of men are just sufficient to form a hollow equilateral wedge, three deep, and if 597 be taken away, the remainder will form a hollow square four deep, the front of which contains one man more than the square root of the number contained in a front of the wedge; what is the number of men?

Ans. 693.

19. Two merchants enter into partnership with 100*l.*; one has his money in business for three months, and the other for two months; and each receives 99*l.* for his capital and profit; find the contribution of each.

Ans. 45*l.* and 55*l.*

20. Two detachments of infantry are ordered to a station distant 39 miles; they begin their march at the same time, but one party by travelling  $\frac{1}{4}$  of a mile an hour more than the other, arrives one hour sooner; required the rates of marching.

Ans. 3 and  $3\frac{1}{4}$  miles an hour.

21. A vintner sold 7 dozen of sherry and 12 dozen of claret for 50*l.*; he sold 3 dozen more of sherry for 10*l.* than he did of claret for 6*l.*; required the price of each.

Ans. Sherry, 2*l.* per dozen; claret, 3*l.* per dozen.

22. The number of men in both fronts of two columns of troops, A and B, when each consisted of as many ranks as it had men in front, was 84; but when the columns changed ground, and A was drawn up with the front B had, and B with the front A had, the number of ranks in both columns was 91; required the number of men in each column.

Ans. 2304 and 1296.

23. A and B lay out some money in speculation; A disposes of his bargain for 11*l.* and gains as much per cent. as B lays out; B's gain is 36*l.*, and it appears that A gains four times as much per cent. as B; required the capital of each.

A's capital, 5*l.*, and B's, 120*l.*

\* 24. A detachment from an army was marching in regular column, with 5 men more in depth than in front; but upon the enemy coming in sight, the front was increased by 845 men, and by this movement the detachment was drawn up in five lines; required the number of men.      Ans. 4550 men.

25. In a mixture of wine and cider, one half, together with 25 gallons, was wine, and the cider was less than one-third part of the mixture by 5 gallons; how many gallons of each did it contain?      Ans. 85 of wine, and 35 of cider.

26. The difference between the hypotenuse and base of a right-angled triangle is = 6, and the difference between the hypotenuse and the perpendicular is = 3; what are the sides?      Ans. 15, 12, and 9.

27. There is a number, consisting of two digits, which being multiplied by the digit on the left hand, the product is 46; but if the sum of the digits be multiplied by the same digit, the product is only 10; required the number.      Ans. 23.

28. A and B gained by trading 100*l.*; half of A's stock was less than B's by 100*l.* and A's gain was three-twentieths of B's stock; what did each put into stock, and what are the respective shares of the gain?

Ans. A's stock was 600*l.*, and B's 400*l.*; A's gain was 60*l.*, B's 40*l.*

29. From two places, at the distance of 320 miles, two persons, A and B, set out at the same time to meet each other; A travelled 8 miles a day more than B, and the number of days until they met was equal to half the number of miles B went in a day; how many miles did each travel per day, and how far did each travel?

Ans. A went 24, B 16 miles per day; A went 192, B 128 miles.

30. There are two rectangular vats, the greater of which contains 20 solid feet more than the other. Their capacities are in the ratio of 4 to 5, and their bases are squares, a side of each of which is equal to the depth of the other; what are the depths?      Ans. 5 feet, and 4 feet.

31. Three persons divide a certain sum of money amongst them in the following manner: A takes the  $n$ th part of the whole, together with  $\frac{a}{n}$ *£*, B takes the  $n$ th part of the remain-

der, together with  $\frac{a}{n}$  £, C takes the  $n$ th part of that which now remains, together with  $\frac{a}{n}$  £, after which nothing remains; find the sum of money.

$$\text{Ans. } \frac{3n^2 - 3n + 1}{(n - 1)^3} \cdot a \text{ £.}$$

32. The distance between two places is  $a$ , and on the first day  $\frac{1}{m}$ th of the journey is performed, on the second day  $\frac{1}{n}$ th of the remainder, then  $\frac{1}{m}$ th and  $\frac{1}{n}$ th of the remainders alternately on succeeding days; find the distance gone over in  $2p$  days.

$$\text{Ans. } a \left\{ 1 - \left( 1 - \frac{1}{m} \right)^p \cdot \left( 1 - \frac{1}{n} \right)^p \right\}.$$

## CHAPTER VII.

### INEQUALITIES, RATIO, PROPORTION, AND VARIATION.

**XVIII.** *Inequalities* are indicated by the sign  $>$  *greater than*, or  $<$  *less than*; thus  $5 > 3$ ,  $4 < 6$ ,  $a > b$ , are inequalities. They may be treated in the same manner as equations, excepting that when *all* the terms have their signs changed, the sign  $>$  must be changed to  $<$ , and the sign  $<$  to  $>$ ; for although  $a$  may be  $> b$ ,  $-a$  is not necessarily  $> -b$ ; for instance,  $5 > 3$ , but  $-5 < -3$ .

#### *Examples.*

1. Let  $m$  and  $n$  be any two *unequal* quantities, then

$$m^2 + n^2 > 2mn.$$

For  $\therefore$  no square quantity can be negative,

$$\therefore (m - n)^2 \text{ or } m^2 + n^2 - 2mn \text{ is positive;}$$

$\therefore$  the positive part of this expression  $>$  the negative part; that is,  $m^2 + n^2 > 2mn$ .

2. Show that  $\sqrt{11} + \sqrt{7}$  is greater than  $\sqrt{19} + \sqrt{2}$ .

$$\sqrt{11} + \sqrt{7} > \text{ or } < \sqrt{19} + \sqrt{2}$$

according as  $(\sqrt{11} + \sqrt{7})^2 > \text{ or } < (\sqrt{19} + \sqrt{2})^2$ ,

$$\text{or, } 18 + 2\sqrt{77} > \text{ or } < 21 + 2\sqrt{38},$$

or,  $2\sqrt{77} > \text{ or } < 3 + 2\sqrt{38}$ , by subtracting 18 from each,

$$\text{or, } 308 > \text{ or } < 161 + 6\sqrt{38}; \text{ by squaring each,}$$

$$\text{or, } 147 > \text{ or } < 6\sqrt{38}, \text{ by subtracting 161 from each.}$$

Now it is evident that 147 is greater than  $6\sqrt{38}$ ,

$$\therefore \sqrt{11} + \sqrt{7} \text{ is greater than } \sqrt{19} + \sqrt{2}.$$

3. Show that every fraction + its reciprocal is  $> 2$ .

Let  $\frac{m}{n}$  be the fraction, then  $\frac{n}{m}$  is its reciprocal.

$$\text{Now } \frac{m}{n} + \frac{n}{m} > \text{ or } < 2$$

$$\text{according as } \left(\frac{m}{n} + \frac{n}{m}\right)^2 > \text{ or } < 2^2,$$

$$\text{or, } \frac{m^2}{n^2} + 2 + \frac{n^2}{m^2} > \text{ or } < 4,$$

$$\text{or, } \frac{m^2}{n^2} + \frac{n^2}{m^2} > \text{ or } < 2.$$

$$\text{But } \therefore 2 \frac{m}{n} \cdot \frac{n}{m} = 2, \quad \therefore \frac{m^2}{n^2} + \frac{n^2}{m^2} > 2, \text{ (by Ex. 1.)}$$

$$\text{Hence } \frac{m}{n} + \frac{n}{m} > 2.$$

4. If  $y = 1 + 4x - x^2$ , what value of  $x$  makes  $y$  the greatest possible?

$$x^2 - 4x = 1 - y,$$

$$x^2 - 4x + 4 = 5 - y, \quad \therefore x - 2 = \sqrt{5 - y},$$

$$\therefore x = 2 \pm \sqrt{5 - y}.$$

Ans.  $x$  must  $= 2$ , and then  $y$  will  $= 5$ ; for the value of  $x$  above found, namely,  $2 \pm \sqrt{5 - y}$ , becomes impossible when  $y > 5$ ; and when  $y$  becomes  $= 5$ , the expression  $\pm \sqrt{5 - y}$  vanishes, and  $x = 2$ .

5. Which is greater,  $\sqrt{7} + \sqrt{10}$ , or  $\sqrt{3} + \sqrt{19}$ ?

Ans.  $\sqrt{3} + \sqrt{19}$ .

6. If  $4x - 7 < 2x + 3$ , and  $3x + 1 > 13 - x$ , find an integral value of  $x$ .

Ans.  $x = 4$ .

7. Show that  $\frac{a+b}{2} > \frac{2ab}{a+b}$ ; and  $a^5 + b^5 > a^4b + ab^4$ .

8. Show that  $\frac{n^2 - n + 1}{n^2 + n + 1}$  lies between 3 and  $\frac{1}{3}$ , for all real values of  $n$ .

## RATIO.

XIX. *Ratio* is the relation which one quantity bears to another with respect to magnitude; thus 12 is 3 times as great as 4, and the ratio of 12 to 4 is 3; this is expressed 12 : 4, the former 12 being called the *antecedent*, and the latter 4 the *consequent* of the ratio.

It is obvious that this ratio might be expressed by the fraction  $\frac{12}{4}$ ,  $\therefore \frac{12}{4} = 3$ .

Similarly the ratio 12 : 36  $= \frac{12}{36} = \frac{1}{3}$ , and in general the

value of the ratio of  $a : b$  may be expressed by  $\frac{a}{b}$ .

The value of a ratio is not altered by multiplying or dividing both its terms by the same quantity.

$$\text{For } a : b = \frac{a}{b} = \frac{na}{nb}, \quad \therefore a : b = na : nb.$$

A ratio is called a ratio of *greater inequality*, of *less inequality*, or of *equality*, according as the antecedent is *greater than*, *less than*, or *equal to*, the consequent.

A ratio of greater inequality is diminished, and a ratio of less inequality is increased, by adding the same quantity to both its terms.

If  $a : b$  be the original ratio, and  $x$  be added to both its terms, the new ratio is  $a + x : b + x$ . Now  $\frac{a + x}{b + x} > \text{or} < \frac{a}{b}$

according as  $\frac{ab + bx}{b(b + x)} > \text{or} < \frac{ab + ax}{b(b + x)}$  by reduction to a common denominator,

$\therefore \frac{a + x}{b + x} > \text{or} < \frac{a}{b}$  according as  $ab + bx > \text{or} < ab + ax$ , according as  $bx > \text{or} < ax$ , according as  $b > \text{or} < a$  (Art. XVIII.); that is,

if  $a > b$ ,  $\frac{a + x}{b + x} < \frac{a}{b}$ , or the ratio is diminished;

if  $a < b$ ,  $\frac{a + x}{b + x} > \frac{a}{b}$ , or the ratio is increased.

A ratio of greater inequality is increased, and a ratio of less inequality is diminished by subtracting the same quantity from both its terms.

If  $x$ , less than  $a$  and  $b$ , be subtracted from them both, the original ratio  $= \frac{a}{b}$ , and the new ratio  $= \frac{a - x}{b - x}$ ;

$$\therefore \frac{a - x}{b - x} > \text{or} < \frac{a}{b} \text{ as } \frac{ab - bx}{b(b - x)} > \text{or} < \frac{ab - ax}{b(b - x)}$$

$$\text{as } ab - bx > \text{or} < ab - ax;$$

whence, adding  $ax + bx$  to both members of the inequality, in order to avoid detached negative quantities,

$$\frac{a - x}{b - x} > \text{or} < \frac{a}{b} \text{ according as}$$

$ab - bx + ax + bx > \text{or} < ab - ax + ax + bx;$   
as,  $ab + ax > \text{or} < ab + bx;$  as  $ax > \text{or} < bx;$  as  $a >$   
or  $< b,$  contrary to the last; that is,

if  $a > b$ ,  $\frac{a-x}{b-x} > \frac{a}{b}$ , or the ratio is increased,

if  $a < b$ ,  $\frac{a-x}{b-x} < \frac{a}{b}$ , or the ratio is diminished.

If the ratios  $a : b$  and  $c : d$  have their antecedents multiplied together, and also their consequents, the resulting ratio  $ac : bd$  is said to be *compounded* of the two former.

The ratio  $a^2 : b^2$  is called the *duplicate* ratio of  $a$  to  $b$ .

$\sqrt{a} : \sqrt{b}$  ..... *subduplicate.* .....

$a^3 : b^3$  ..... triplicate. ....

$$\sqrt[3]{a} : \sqrt[3]{b} \dots\dots\dots \textit{subtriplicate.} \dots\dots\dots$$

&c. &c.

## PROPORTION.

XX. *Proportion* is the equality of ratios. Thus if the ratio  $a : b$  be equal to the ratio  $c : d$ , that is, if  $\frac{a}{b} = \frac{c}{d}$ , the four quantities  $a, b, c, d$  are called *proportionals*, and  $a : b :: c : d$ , or  $a : b = c : d$ , is called a *proportion*.

### Propositions.

1. If four quantities are proportionals, the product of the *extremes* is equal to the product of the *means*.

Let  $a : b :: c : d$  be the proportion,

then  $\frac{a}{b} = \frac{c}{d}$ , and, multiplying both members of this equation by  $bd$ , we have  $ad = bc$ .

*Cor. 1.* If  $a : b :: b : c$ , then  $ac = b^2$ , and  $\therefore b = \sqrt{ac}$ , where  $b$  is a mean proportional to  $a$  and  $c$ . Hence a mean proportional between any two quantities is equal to the square root of their product.

Cor. 2.  $\because ad = bc, \therefore a = \frac{bc}{d}, b = \frac{ad}{c}, c = \frac{ad}{b}, d = \frac{bc}{a};$



Hence, if *three* terms of a proportion be given, the *fourth* may be found. The Rule of Three, in Arithmetic, is derived from this.

2. If the product of two quantities be equal to the product of two others, the four will constitute a proportion if the terms of one product be made the extremes, and the terms of the other the means.

Let  $nx = my$ , then, dividing by  $ny$ ,

$$\frac{x}{y} = \frac{m}{n}, \quad \therefore x : y :: m : n.$$

3. If  $a : b :: c : d$ , and  $c : d :: m : n$ , then  $a : b :: m : n$ . (Euclid, Book V. Prop. 11.)

$$\text{For } \frac{a}{b} = \frac{c}{d}, \text{ and } \frac{c}{d} = \frac{m}{n}, \quad \therefore \frac{a}{b} = \frac{m}{n}, \\ \therefore a : b :: m : n.$$

4. If  $a : b :: c : d$ , then  $b : a :: d : c$ . (Euclid, Book V. Prop. B.)

$$\text{For } \frac{a}{b} = \frac{c}{d}, \quad \therefore \frac{b}{a} = \frac{d}{c}, \quad \therefore b : a :: d : c.$$

This operation is called *invertendo*.

5. If  $a : b :: c : d$ , then  $a : c :: b : d$ . (Euclid, Book V. Prop. 16.)

$$\text{For } \frac{a}{b} = \frac{c}{d}, \text{ and, multiplying by } \frac{b}{c}, \text{ we have}$$

$$\frac{a}{c} = \frac{b}{d}, \quad \therefore a : c :: b : d.$$

This is called *alternando*.

6. If  $a : b :: c : d$ , then  $a + b : b :: c + d : d$ . (Euclid, Book V. Prop. 18.)

$$\text{For } \frac{a}{b} = \frac{c}{d}, \text{ and, adding 1 to each side,}$$

$$\frac{a}{b} + 1 = \frac{c}{d} + 1, \quad \text{or } \frac{a + b}{b} = \frac{c + d}{d},$$

$$\therefore a + b : b :: c + d : d.$$

This is called *componendo*.

7. If  $a : b :: c : d$ , then  $a - b : b :: c - d : d$ . (Euclid, Book V. Prop. 17.)

$$\text{For } \frac{a}{b} = \frac{c}{d}, \therefore \frac{a}{b} - 1 = \frac{c}{d} - 1,$$

$$\text{or } \frac{a-b}{b} = \frac{c-d}{d}, \therefore a-b : b :: c-d : d.$$

This is called *dividendo*.

8. If  $a : b :: c : d$ , then  $a - b : a :: c - d : c$ ,  
and  $a : a - b :: c : c - d$ . (Euclid, Book V. Prop. E.)

$$\text{For } \frac{a-b}{b} = \frac{c-d}{d}, \text{ by Prop. 7,}$$

$$\text{and } \frac{b}{a} = \frac{d}{c}, \quad \dots \quad 4,$$

$$\therefore \frac{a-b}{b} \times \frac{b}{a} = \frac{c-d}{d} \times \frac{d}{c}, \text{ or } \frac{a-b}{a} = \frac{c-d}{c},$$

$$\therefore a-b : a :: c-d : c,$$

$$\text{and } a : a-b :: c : c-d, \text{ invertendo.}$$

9. If  $a : b :: c : d$ , then  $a + b : a - b :: c + d : c - d$ .

$$\text{For } \frac{a+b}{b} = \frac{c+d}{d}, \text{ by Prop. 6,}$$

$$\text{and } \frac{a-b}{b} = \frac{c-d}{d}, \quad \dots \quad 7,$$

$$\therefore \frac{a+b}{b} \div \frac{a-b}{b} = \frac{c+d}{d} \div \frac{c-d}{d}, \text{ or } \frac{a+b}{a-b} = \frac{c+d}{c-d},$$

$$\therefore a+b : a-b :: c+d : c-d.$$

10. If  $a : b :: c : d :: e : f$ , &c., then

$a : b :: a + c + e + \&c. : b + d + f + \&c.$  (Euclid, Book V. Prop. 12.)

$$\therefore \frac{a}{b} = \frac{c}{d}, \therefore ad = bc, \therefore \frac{a}{b} = \frac{e}{f}, \therefore af = be, \text{ and } ab = ba.$$

Hence  $ab + ad + af = ba + bc + be$ , by addition,

or  $a(b + d + f) = b(a + c + e)$ ,

$\therefore a : b :: a + c + e : b + d + f$ , by Prop. 2,  
and similarly when more quantities are taken.

11. If  $a : b :: c : d$ , then  $ma : mb :: \frac{c}{n} : \frac{d}{n}$ ,

and  $ma : \frac{b}{n} :: mc : \frac{d}{n}$ .

For  $\frac{a}{b} = \frac{c}{d}$ ,  $\therefore \frac{ma}{mb} = \frac{\frac{c}{n}}{\frac{d}{n}}$  (Art. IX.),

$\therefore ma : mb :: \frac{c}{n} : \frac{d}{n}$ .

Again,  $\frac{a}{b} = \frac{c}{d}$ ,  $\therefore \frac{ma}{b} = \frac{mc}{d}$ ,  $\frac{ma}{\frac{b}{n}} = \frac{mc}{\frac{d}{n}}$ ,

$\therefore ma : \frac{b}{n} :: mc : \frac{d}{n}$ .

12. If  $a : b :: c : d$   
and  $e : f :: g : h$  } , then  $ae : bf :: cg : dh$ ,

For  $\frac{a}{b} = \frac{c}{d}$ ,  $\frac{e}{f} = \frac{g}{h}$ ,  $\therefore \frac{ae}{bf} = \frac{cg}{dh}$ , by multiplication,

$\therefore ae : bf :: cg : dh$ .

and similarly for any number of proportions.

13. If  $a : b :: c : d$ , then  $a^n : b^n :: c^n : d^n$ ,  $n$  being either integral or fractional.

For  $\frac{a}{b} = \frac{c}{d}$ ,  $\therefore \frac{a^n}{b^n} = \frac{c^n}{d^n}$ ,  $\therefore a^n : b^n :: c^n : d^n$ .

14. If  $a : b :: b : c$ , then  $a : c :: a^2 : b^2$ , (Euclid, Book V. Def. 10.)

For  $\frac{a}{b} = \frac{b}{c}$ ,  $\therefore \frac{a}{c} = \frac{a}{b} \times \frac{b}{c} = \frac{a}{b} \times \frac{a}{b} = \frac{a^2}{b^2}$ ,  
 $\therefore a : c :: a^2 : b^2$ .

15. If  $a : b :: b : c :: c : d$ , then  $a : d :: a^3 : b^3$ . (Euclid, Book V. Def. 11.)

$$\text{For } \frac{a}{d} = \frac{a}{b} \times \frac{b}{c} \times \frac{c}{d}, \text{ and } \frac{a}{b} = \frac{b}{c} = \frac{c}{d},$$

$$\therefore \frac{a}{d} = \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} = \frac{a^3}{b^3}, \therefore a : d :: a^3 : b^3.$$

In this proposition, and the preceding, the given quantities are in *continued proportion*.

## VARIATION.

XXI. When two quantities have such a mutual relation that one being changed the other is changed in the *same proportion*, they are said to *vary directly* as each other.

Suppose A and B to have such a relation to each other, that if the value of A becomes  $a$ , B will have such a value  $b$ , as that  $A : a :: B : b$ ; then A varies directly as B, or  $A \propto B$ .

For instance, let  $\Delta$  represent the area of a triangle,  $b$  the base, and  $p$  the perpendicular altitude; then if  $p$  be constant,  $\Delta \propto b$ . (Euclid, Book VI. Prop. 1.)

Suppose A and B to have such a relation to each other, that A being changed to  $a$ , B is changed to  $b$  in such a manner that  $A : a :: \frac{1}{B} : \frac{1}{b}$ ; then A varies inversely as B, or  $A \propto \frac{1}{B}$ .

Let  $\Delta$ ,  $b$  and  $p$  represent the same magnitudes as before, then  $\Delta = \frac{1}{2}bp$  (Euclid, Book I. Prop. 41.); and if  $\Delta$  is constant, and  $b$  increases,  $p$  must decrease in the same proportion, thus if  $b$  becomes  $mb$ ,  $p$  will become  $\frac{p}{m}$ ,  $\therefore b$  varies *inversely* as  $p$ , or  $b \propto \frac{1}{p}$ .

*Prop.* If A varies as B, A is equal to B multiplied by some invariable quantity.

For  $\because A : a :: B : b, \therefore A : a :: mB : mb, \therefore A : mB :: a : mb$  alternando; if, therefore,  $m$  be so taken that  $A = mB$ , then, in all cases  $a = mb$ .

Conversely, if  $A = mB$ , and  $m$  is constant, then  $A \propto B$ .

\* \* This proposition is of very extensive use in Variation, since the variation is convertible into an equation.

*Examples in Inequalities, Ratio, Proportion, and Variation.*

1. Which is greater, the ratio of 15 : 17 or 17 : 19?

$$\left. \begin{aligned} \frac{15}{17} &= \frac{15}{17} \times \frac{19}{19} = \frac{285}{323} \\ \frac{17}{19} &= \frac{17}{19} \times \frac{17}{17} = \frac{289}{323} \end{aligned} \right\} \therefore 17 : 19 \text{ is greater.}$$

2. Prove that  $a^3 + b^3 : a^2 + b^2$  is greater than  $a^3 + b^3 : a + b$ .

$$\frac{a^3 + b^3}{a^2 + b^2} = \frac{a^3 + b^3}{a^2 + b^2} \times \frac{a + b}{a + b} = \frac{a^4 + a^3b + ab^3 + b^4}{(a^2 + b^2)(a + b)},$$

$$\frac{a^3 + b^3}{a + b} = \frac{a^3 + b^3}{a + b} \times \frac{a^2 + b^2}{a^2 + b^2} = \frac{a^4 + 2a^2b^2 + b^4}{(a^2 + b^2)(a + b)},$$

$$\therefore a^3 + b^3 : a^2 + b^2 > \text{ or } < a^3 + b^3 : a + b$$

according as  $a^3b + ab^3 > \text{ or } < 2a^2b^2$ ,

or, as  $a^2 + b^2 > \text{ or } < 2ab$ ,

$\therefore a^3 + b^3 : a^2 + b^2 > a^3 + b^3 : a + b$ . (Art. XVIII. Ex. 1.)

3. If  $x : y$  in the duplicate ratio of  $a : b$ , and  $a : b$  in the sub-duplicate ratio of  $a + x : a - y$ , then  $2x : a :: x - y : y$ .

$$x : y :: a^2 : b^2, \dots\dots\dots (1) \quad \therefore \frac{x}{y} = \frac{a^2}{b^2},$$

$$\sqrt{a + x} : \sqrt{a - y} :: a : b, (2) \quad \therefore \frac{a + x}{a - y} = \frac{a^2}{b^2},$$

$$\therefore \frac{a+x}{a-y} = \frac{x}{y}, \quad \therefore a+x : a-y :: x : y.$$

$$x+y : a-y :: x-y : y \dots \text{dividendo, (Art. XX.)}$$

$$x+y : x-y :: a-y : y \dots \text{alternando,}$$

$$2x : x-y :: a : y \dots \text{componendo,}$$

$$\therefore 2x : a :: x-y : y \dots \text{alternando.}$$

$$4. \text{ If } x^2 + y^2 : xy :: 13 : 6, \text{ and } x^2 - y^2 = 20; \ x = 6, y = 4.$$

$$\text{By Art. XX., } x^2 + y^2 : 2xy :: 13 : 12,$$

$$x^2 + 2xy + y^2 : x^2 - 2xy + y^2 :: 25 : 1,$$

$$x+y : x-y :: 5 : 1,$$

$$5x - 5y = x + y, \quad \therefore 4x = 6y, \quad \therefore x = \frac{3}{2}y;$$

$$\frac{9}{4}y^2 - y^2 = 20, \quad 9y^2 - 4y^2 = 80,$$

$$5y^2 = 80, \quad \therefore y^2 = 16, \quad \therefore y = \pm 4,$$

$$x = \frac{3}{2}y = \frac{3}{2} \times \pm 4 = \pm 6.$$

5. If  $y = p + q + r$ , where  $p$  is constant,  $q$  varies as  $x$ , and  $r$  as  $\frac{1}{x}$ , and if, when  $x = 1, 2, 3$ ,  $y = 3, 5\frac{1}{2}, 7$ ; show that  $y = 5 + x - \frac{3}{x}$ .

$$\text{Let } q = ax, r = \frac{b}{x}, \quad \therefore y = p + ax + \frac{b}{x}.$$

$$\text{But if } x = 1, y = 3, \quad \therefore 3 = p + a + b \dots \dots \dots (1)$$

$$\dots \ x = 2, y = \frac{11}{2}, \quad \frac{11}{2} = p + 2a + \frac{b}{2} \dots \dots \dots (2)$$

$$\dots \ x = 3, y = 7, \quad 7 = p + 3a + \frac{b}{3} \dots \dots \dots (3)$$

$$\left. \begin{aligned} (2) - (1), \quad \frac{5}{2} &= a - \frac{b}{2} \\ (3) - (2), \quad \frac{3}{2} &= a - \frac{b}{6} \end{aligned} \right\}$$

$$1 = -\frac{b}{3}, \quad \therefore b = -3,$$

$$\frac{5}{2} = a + \frac{3}{2}, \quad \therefore a = 1.$$

$$3 = p + 1 - 3, \quad \therefore p = 5,$$

$$\therefore y = 5 + x - \frac{3}{x}.$$

6. If  $x^3 = a^3 + b^3$ , and  $y^3 = c^3 + d^3$ , show that

$xy$  is greater than  $ac + bd$ , and  $ad + bc$ .

$$\begin{aligned} x^2y^2 &= a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2, \\ &= a^2c^2 + 2acbd + b^2d^2 + a^2d^2 - 2adb c + b^2c^2, \\ &= (ac + bd)^2 + (ad - bc)^2, \end{aligned}$$

$$\therefore xy = \sqrt{(ac + bd)^2 + (ad - bc)^2}, \text{ which is } > ac + bd.$$

$$\begin{aligned} \text{Again, } x^2y^2 &= a^2d^2 + b^2c^2 + a^2c^2 + b^2d^2, \\ &= a^2d^2 + 2adb c + b^2c^2 + a^2c^2 - 2acbd + b^2d^2, \\ &= (ad + bc)^2 + (ac - bd)^2, \end{aligned}$$

$$\therefore xy = \sqrt{(ad + bc)^2 + (ac - bd)^2}, \text{ which is } > ad + bc.$$

7. If  $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \&c.$ , prove that

$$(1) \frac{a}{b} = \sqrt{\frac{a^2 + b^2 + c^2 + \&c.}{b^2 + d^2 + f^2 + \&c.}}, \quad (2) \frac{a}{b} = \frac{ma + nc + pc + \&c.}{mb + nd + pf + \&c.}.$$

$$(1) \quad \frac{a}{b} = \frac{c}{d}, \quad \therefore ad = bc, \quad \therefore a^2 d^2 = b^2 c^2,$$

$$\frac{a}{b} = \frac{e}{f}, \quad \therefore af = be, \quad \therefore a^2 f^2 = b^2 e^2,$$

Hence, by addition,  $a^2 d^2 + a^2 f^2 = a^2 b^2 + b^2 c^2 + b^2 e^2$ ,  
 or,  $a^2 (d^2 + f^2) = b^2 (a^2 + c^2 + e^2)$ ,

$$\therefore \frac{a^2}{b^2} = \frac{a^2 + c^2 + e^2}{b^2 + d^2 + f^2}, \quad \therefore \frac{a}{b} = \sqrt{\frac{a^2 + c^2 + e^2}{b^2 + d^2 + f^2}} + \&c.$$

$$(2) \quad \frac{a}{b} = \frac{ma}{mb}, \quad \frac{a}{b} = \frac{c}{d}, \quad \frac{a}{b} = \frac{e}{f},$$

$$ad = bc, \quad af = be,$$

$$amb = bma, \quad and = bnc, \quad apf = bpe.$$

Hence,  $amb + and + apf = bma + bnc + bpe$ ,

or,  $a(mb + nd + pf) = b(ma + nc + pe)$ ,

$$\therefore \frac{a}{b} = \frac{ma + nc + pe}{mb + nd + pf} + \&c.$$

8. The ratio of the sum of the sides of a right-angled triangle to the hypotenuse is  $m : n$ ; show that  $m$  cannot be greater than  $n\sqrt{2}$ .

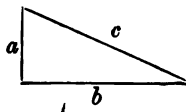
Let  $a$  and  $b$  be the sides,  $c$  the hypotenuse, then  $a + b : c :: m : n$ ,

$a^2 + b^2 = c^2$ , (Euclid, Book I. Prop. 47.)

$$\frac{a+b}{c} = \frac{m}{n}, \quad \therefore a+b = c \cdot \frac{m}{n},$$

$$\left. \begin{array}{l} a^2 + 2ab + b^2 = c^2 \cdot \frac{m^2}{n^2} \\ 2a^2 + 2b^2 = 2c^2 \end{array} \right\} \text{ Subtract.}$$

$$a^2 - 2ab + b^2 = 2c^2 - c^2 \cdot \frac{m^2}{n^2} = \frac{c^2}{n^2} (2n^2 - m^2),$$





$$\left. \begin{aligned} \therefore a - b &= \frac{c}{n} \sqrt{2n^2 - m^2} \\ a + b &= \frac{c}{n} \cdot m \end{aligned} \right\} \text{Add and subtract.}$$

$$\therefore 2a = \frac{c}{n} (m \pm \sqrt{2n^2 - m^2}),$$

$$2b = \frac{c}{n} (m \mp \sqrt{2n^2 - m^2}),$$

Now these values of  $2a$  and  $2b$  become impossible when  $m^2 > 2n^2$ , or  $m > n\sqrt{2}$ ,

$\therefore m$  cannot be greater than  $n\sqrt{2}$ .

### Examples.

1. Which is greater,  $3 : 7$  or  $4 : 9$ ? Ans.  $4 : 9$ .
2. Compound the ratios  $2 : 9$  and  $12 : 5$ . Ans.  $8 : 15$ .
3. Compound the ratios  $m : 6x^2$ ,  $8y^2 : n$ ,  $a^2 : 2y^2$ .  
Ans.  $m : 4n$ .
4. If  $x > y$  which is greater,  $x - y$  or  $(x^{\frac{1}{2}} - y^{\frac{1}{2}})^2$ ?
5. If  $A \propto B$ , and  $B \propto C$ , show that  $A \propto C$ .
6. If  $a : b :: c : d$ , prove that  
 $ma \pm nb : pa \pm qb :: ma \pm nd : pc \pm qd$ .
7. If  $x = y + 12$ , and  $\sqrt{xy} : 4 :: \frac{1}{2}(x + y) : 5$ , show that  $x = 16$  and  $y = 4$ .
8. If  $a > b$ , show that  $\sqrt{a^2 - b^2} + \sqrt{a^2 - (a - b)^2} > a$ .
9.  $y \propto x$ , and when  $x = 2$ ,  $y = 10$ , show that  $y = 5x$ .
10. Find two numbers in the ratio of 3 to 2, whose sum multiplied by their product is equal to 12 times the difference of their squares.  
Ans. 6 and 4.
11. Let  $y^2 \propto a^2 - x^2$ , and when  $x = 0$  suppose  $y = b$ ; then shall  $y^2 = \frac{b^2}{a^2}(a^2 - x^2)$ .

12. If  $x : y :: a^3 : b^3$ , and  $\sqrt[3]{c+x} : \sqrt[3]{d+y} :: a : b$ , how that  $dx = cy$ .

13. The sum of three numbers in continued proportion is 2, and the sum of the extremes : the mean :: 10 : 3. Find them.  
Ans. 4, 12, 36.

## CHAPTER VIII.

### ARITHMETICAL PROGRESSION.

XXII. When the terms of a series of quantities *increase* or *decrease* by a common difference, the series is called an *Arithmetical Progression*. Thus 1, 3, 5, 7, 9 is an arithmetical progression whose first term is 1, last term 9, common difference 2, number of terms 5, and sum 25. If this series were written thus, 9, 7, 5, 3, 1, its first term would be 9, last term 1, and common difference  $-2$ .

*Prop.* To find the  $n$ th term and the sum of any arithmetical progression.

Let  $a$  be the first term,  $l$  the last or  $n$ th term,  $d$  the common difference,  $n$  the number of terms, and  $S$  the sum of the series: then the series will be

$$a, (a + d), (a + 2d), (a + 3d), \dots, \{a + (n - 1)d\}.$$

Now any term of this series = the first term + the common difference multiplied by a factor less by unity than the place of the term; and the last term  $l = a + (n - 1)d$ . The same series written backwards would be

$$l, (l - d), (l - 2d), (l - 3d), \dots, a.$$

$$\text{Hence } S = a + (a + d) + (a + 2d) \dots + (l - 2d) + (l - d) + l,$$

$$\text{and } S = l + (l - d) + (l - 2d) \dots + (a + 2d) + (a + d) + a,$$

$$\therefore 2S = (l + a) + (l + a) + (l + a) \dots + (l + a) + (l + a) + (l + a) \\ = (l + a) \text{ repeated } n \text{ times,}$$

$$\therefore 2S = (l + a) \cdot n, \quad \therefore S = (l + a) \cdot \frac{n}{2}.$$

$$\text{But } \because l = a + (n - 1)d, \quad \therefore l + a = 2a + (n - 1)d,$$

$$\therefore S = \{2a + (n - 1)d\} \cdot \frac{n}{2} = na + \frac{n(n - 1)d}{2}.$$

An *arithmetic mean* between two quantities = half their sum; thus if  $m = \frac{a+b}{2}$ ,  $m$  is an arithmetic mean between  $a$  and  $b$ .

When the number of terms of an arithmetic series is *odd* the middle term = half the sum of any two terms equidistant from it.

*Examples.*

1. Find the sum of  $2 + 5 + 8 + \&c.$ , to 17 terms.

Here  $a = 2$ ,  $d = 3$ ,  $n = 17$ ,

$$\begin{aligned}\therefore S &= na + \frac{n(n-1)d}{2} = 17 \times 2 + \frac{17 \times 16 \times 3}{2} \\ &= 34 + 408 = 442. \quad \text{Ans.}\end{aligned}$$

2. Find the sum of  $\frac{1}{2} + \frac{3}{8} + \frac{1}{4} + \&c.$ , to 20 terms.

Here  $a = \frac{1}{2}$ ,  $d = -\frac{1}{8}$ ,  $n = 20$ ,

$$\begin{aligned}\therefore S &= na + \frac{n(n-1)d}{2} = 20 \times \frac{1}{2} + \frac{20 \times 19 \times -\frac{1}{8}}{2}, \\ &= 10 - \frac{95}{4} = \frac{40 - 95}{4} = -\frac{55}{4}. \quad \text{Ans.}\end{aligned}$$

3. The sum of an arithmetic series is 950, the common difference 3, and the number of terms 25. What is the last term?

Here  $S = 950$ ,  $d = 3$ ,  $n = 25$ ,

$$S = na + \frac{n(n-1)d}{2},$$

$$\therefore 950 = 25a + \frac{25 \times 24 \times 3}{2} = 25a + 900,$$

$$\therefore 50 = 25a, \quad \therefore a = 2, \text{ the first term.}$$

$$\text{Now } l = a + (n-1)d = 2 + 24 \times 3 = 74. \quad \text{Ans.}$$

4. Insert 5 arithmetic means between 1 and  $-1$ .

Here  $a = 1$ ,  $l = -1$ , and  $\therefore$  there are 5 mean and 2 extreme terms,  $n = 5 + 2 = 7$ .

$$= a + (n-1)d, \therefore d = \frac{l-a}{n-1} = \frac{-1-1}{7-1} = -\frac{2}{6} = -\frac{1}{3}.$$

Hence  $1 - \frac{1}{3} = \frac{2}{3} =$  the first mean.

$$\frac{2}{3} - \frac{1}{3} = \frac{1}{3} = \dots \text{second } \dots$$

$$\frac{1}{3} - \frac{1}{3} = 0 = \dots \text{third } \dots$$

$$0 - \frac{1}{3} = -\frac{1}{3} = \dots \text{fourth } \dots$$

$$-\frac{1}{3} - \frac{1}{3} = -\frac{2}{3} = \dots \text{fifth } \dots$$

Ans.

5. A body of soldiers is drawn up in the form of a hollow quadrilateral wedge, the ranks of which are 3 deep, and the outer rank consisting of  $n$  soldiers. Find the number of soldiers.

First, to find the number, supposing the wedge solid,

$$a = 1, \quad l = n, \quad n = n,$$

$$S = (a + l) \frac{n}{2} = (1 + n) \frac{n}{2} = \frac{n^2 + n}{2}.$$

Next, to find the number that might stand in the hollow part.

$$a = 1, \quad n_1 = n - 9 = l,$$

$$S_1 = (a + l) \frac{n_1}{2} = (1 + n - 9) \cdot \frac{n - 9}{2} = \frac{n^2 - 17n - 72}{2}.$$

Now the former number — the latter = the number sought;

$$\therefore S - S_1 = \frac{n^2 + n}{2} - \frac{n^2 - 17n - 72}{2} = \frac{18n - 72}{2} = 9n - 36.$$

If the ranks be  $r$  deep,  $n_1 = n - 3r$ ; and the number of soldiers  $= \frac{3r}{2} (2n + 1 - 3r)$ .

6. Find the sum of the series  $1^3 + 2^3 + 3^3 \dots + n^3$ .

$$1^3 + 2^3 + 3^3 + \dots + n^3 = 1 + (3 + 5) + (7 + 9 + 11) + (13 + 15 + 17 + 19) \dots + n^3,$$

of which there are  $n$  sets, and the  $n$ th set contains  $n$  terms.

Hence to find the entire number of terms.

$$a = 1, \quad d = 1, \quad n = n,$$

$$\therefore \{2 + n - 1\} \cdot \frac{n}{2} = \frac{n^2 + n}{2} = \text{the number of terms.}$$

Now, to find the sum of the whole series,

$$a = 1, \quad d = 2, \quad n_1 = \frac{n^2 + n}{2},$$

$$\therefore S = \left\{ 2 + \left( \frac{n^2 + n}{2} - 1 \right) \cdot 2 \right\} \cdot \frac{n^2 + n}{4} = \left( \frac{n^2 + n}{2} \right)^2.$$

$$\text{Since } \frac{n^2 + n}{2} = 1 + 2 + 3 \dots + n,$$

$$\therefore 1^3 + 2^3 + 3^3 \dots + n^3 = (1 + 2 + 3 \dots + n)^2.$$

7.  $S_1, S_2, \&c., S_p$  are the sums of  $p$  arithmetical progressions, each continued to  $n$  terms; the first terms are 1, 2, &c., and the common differences 1, 3, 5, &c.: prove that

$$S_1 + S_2 + \&c. + S_p = \frac{1}{2} np(np + 1).$$

$$\left. \begin{array}{l} S_1 = 1, 2, 3 \dots \\ S_2 = 2, 5, 8 \dots \\ S_3 = 3, 8, 13 \dots \end{array} \right\} \&c., \text{ to } n \text{ terms, } \therefore \left\{ \begin{array}{ll} \text{1st com. dif. } 1 = 2 \times 1 - 1 \\ \text{2nd } \dots \quad 3 = 2 \times 2 - 1 \\ \text{3rd } \dots \quad 5 = 2 \times 3 - 1 \\ \dots \quad \dots \quad \dots \\ \text{pth } \dots \quad = 2 \times p - 1 \end{array} \right.$$

$\therefore S_p = p, 3p - 1, 5p - 2, \&c.,$  to  $n$  terms = the  $p$ th series.

Now

$$S_1 = na + \frac{n(n-1)d}{2} = n + \frac{n(n-1)}{2} \cdot 1 = \frac{n^2 + n}{2},$$

$$S_2 = \dots = 2n + \frac{n(n-1)}{2} \cdot 3 = \frac{3n^2 + n}{2},$$

$$S_3 = \dots = 3n + \frac{n(n-1)}{2} \cdot 5 = \frac{5n^2 + n}{2},$$

$$S_p = \dots = pn + \frac{n(n-1)}{2} \cdot (2p-1) = \frac{(2p-1)n^2 + n}{2},$$

the com. diff. of which is  $n^2$ ,  $\therefore$  they are in arith. prog., and their sum is the sum sought.

$$a = \frac{n^2 + n}{2}, \quad l = \frac{(2p-1)n^2 + n}{2}, \quad n = p,$$

$$s = (a + l) \frac{n}{2} = \left\{ \frac{n^2 + n}{2} + \frac{(2p - 1)n^2 + n}{2} \right\} \cdot \frac{p}{2} = \frac{1}{2} np(np + 1).$$

8. A party of foot begin their march at 6 in the morning, and travel  $3\frac{1}{2}$  miles an hour; 3 hours after a troop of horse follow them from the same place, and travel  $3\frac{1}{2}$  miles the first hour, 4 miles the next,  $4\frac{1}{2}$  the third, and so on. In what time will they overtake the foot?

Let  $x$  be the number of hours in which the horse overtake the foot,

$$\text{then for the horse, } a = \frac{7}{2}, \quad d = \frac{1}{2}, \quad n = x,$$

$$\begin{aligned} s &= \frac{n}{2} \{2a + (n - 1)d\} = \frac{x}{2} \left\{ 7 + (x - 1) \cdot \frac{1}{2} \right\} \\ &= \frac{x}{2} \left\{ 7 + \frac{x}{2} - \frac{1}{2} \right\} = \frac{x}{2} \left\{ \frac{13}{2} + \frac{x}{2} \right\} \\ &= \frac{13x + x^2}{4} = \text{miles the horse march.} \end{aligned}$$

Again,  $x + 3 =$  hours the foot march,

$$\frac{7}{2}(x + 3) = \text{miles the foot march,}$$

$$\therefore \frac{13x + x^2}{4} = \frac{7(x + 3)}{2}, \text{ or } 13x + x^2 = 14x + 42,$$

$$x^2 - x = 42, \quad \therefore x = 7 \text{ hours. Ans.}$$

9. Prove that the latter half of  $2n$  terms of an arithmetical series  $= \frac{1}{2}$  of the sum of  $3n$  terms of the same series.

Let  $a, a + d, a + 2d, \&c.,$  be the series.

Then, to find the sum of the latter half of  $2n$  terms,

$a = a, d = d, n = n, l = a + (n - 1)d,$  the  $n$ th or last term of the first half,

$\therefore a + (n - 1)d + d,$  or  $a + nd$  is the first term of the 2nd half, and  $a_1 = a + nd, d = d, n = n.$

$$\begin{aligned}\text{Hence } S &= \frac{n}{2} \{2a + (n-1)d\} = \frac{n}{2} \{2(a+nd) + (n-1)d\} = \\ &= \frac{n}{2} \{2a + 2nd + nd - d\} = \frac{n}{2} \{2a + 3nd - d\} = \\ &\quad \text{sum of latter half of } 2n \text{ terms.}\end{aligned}$$

Now to find the sum of  $3n$  terms,

$$a = a, \quad d = d, \quad n_1 = 3n,$$

$$\begin{aligned}\therefore S_1 &= \frac{n_1}{2} \{2a + (n_1-1)d\} = \frac{3n}{2} \{2a + (3n-1)d\} = \\ &= \frac{3n}{2} \{2a + 3nd - d\},\end{aligned}$$

$$\therefore \frac{n}{2} \{2a + 3nd - d\} \text{ is } \frac{1}{3} \text{ of the sum of } 3n \text{ terms.}$$

Hence, the latter half of  $2n$  terms  $= \frac{1}{3}$  the sum of  $3n$  terms.

10. If  $x^2 + (x+1)^2 + (x+2)^2 + \dots$  to 9 terms  $= 501$ , find  $x$ .

Let  $z-4 = x$ , then the series becomes

$$(z-4)^2 + (z-3)^2 + (z-2)^2 + (z-1)^2 + z^2 + (z+1)^2 + (z+2)^2 + (z+3)^2 + (z+4)^2 = 501,$$

$$2z^2 + 32 + 2z^2 + 18 + 2z^2 + 8 + 2z^2 + 2 + z^2 = 501,$$

$$9z^2 + 60 = 501, \quad 9z^2 = 441, \quad 3z = 21,$$

$$z = 7, \quad \therefore x = 3.$$

11. The sum of 3 numbers in arithmetical progression is 30, and the sum of their squares is 308; find them.

Let  $x-y$ ,  $x$ ,  $x+y$ , be the numbers,

then  $x-y+x+x+y=30$ , or  $3x=30$ ,  $\therefore x=10$ .

$$x^2 - 2xy + y^2 + x^2 + x^2 + 2xy + y^2 = 308,$$

$$3x^2 + 2y^2 = 308, \text{ or } 2y^2 = 308 - 3x^2 =$$

$$308 - 300 = 8, \quad \therefore y^2 = 4, \quad y = 2 = \text{com. diff.}$$

$$\left. \begin{array}{l} \text{Hence } 10 - 2 = 8. \\ \quad \quad \quad 10. \\ \quad \quad \quad 10 + 2 = 12. \end{array} \right\} \text{Ans.}$$

12. There are 4 numbers in arithmetical progression, whose sum is 24, and product 945; find them.

Let  $x - 3y$ ,  $x - y$ ,  $x + y$ ,  $x + 3y$  be the numbers,

then the sum  $4x = 24$ ,  $\therefore x = 6$ ,

the product  $(x - 3y)(x + 3y)(x - y)(x + y) = 945$ ,

$$\text{or } (x^2 - 9y^2)(x^2 - y^2) = 945,$$

$$\text{or } x^4 - 10x^2y^2 + 9y^4 = 945,$$

$$\therefore \text{ by substitution, } 1296 - 360y^2 + 9y^4 = 945.$$

$$9y^4 - 360y^2 = -351, \therefore y = 1.$$

$$\begin{array}{l} \text{Hence } x - 3y = 6 - 3 = 3. \\ \quad x - y = 6 - 1 = 5. \\ \quad x + y = 6 + 1 = 7. \\ \quad x + 3y = 6 + 3 = 9. \end{array} \quad \left. \vphantom{\begin{array}{l} x - 3y = 6 - 3 = 3. \\ x - y = 6 - 1 = 5. \\ x + y = 6 + 1 = 7. \\ x + 3y = 6 + 3 = 9. \end{array}} \right\} \text{Ans.}$$

### Examples.

1. Find the sum of  $n$  terms of  $1 + 2 + 3 + 4 + \&c.$

$$\text{Ans. } \frac{1}{2}n(n+1).$$

2. The sum of  $9 + 15 + 21, \&c.$ , to 10 terms  $= 360$ .

3. Find the 20th term, and the sum of 20 terms of  $\frac{1}{2} + \frac{1}{3} + \frac{1}{6} + \&c.$

$$\text{Ans. } -2\frac{2}{3}, \text{ and } -21\frac{2}{3}.$$

4. Sum  $1 + 8 + 15 + \&c.$ , to 100 terms. Ans. 34750.

5. Sum  $\frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \&c.$ , to 12 terms. Ans.  $-1\frac{1}{2}$ .

6. The 9th term of the series,  $7, 5\frac{1}{2}, 4, \&c.$   $= -5$ .

7. Required the 24th term, and the sum of 24 terms, of  $\frac{7}{12} + \frac{2}{3} + \frac{3}{4} + \&c.$

$$\text{Ans. } 2\frac{1}{2}, \text{ and } 37.$$

8. Given  $a = 3$ ,  $l = 17$ ,  $n = 29$ ; find the series.

$$\text{Ans. } 3, 3\frac{1}{2}, 4, 4\frac{1}{2}, \&c.$$

9. The 100th term of the series  $1, 9, 17, \&c.$   $= 793$ .

10. The sum of  $\frac{1}{2} + \frac{3}{4} + 1 + \&c.$ , to 10 terms  $= 16\frac{1}{4}$ .

11. Find the sum of the series  $198, 193, 188, \&c.$ , to 40 terms.

$$\text{Ans. } 4020.$$



12. The 20th term of the series  $\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \&c. = 5\frac{5}{6}$ .

13. The sum of  $\frac{1}{2} - \frac{2}{3} - \frac{11}{6} - \&c.$ , to 13 terms  $= -84\frac{1}{2}$ .

14. Sum  $\frac{5}{7} + 1\frac{2}{7} + \&c.$ , to  $n$  terms.      Ans.  $\frac{n}{7}(n+4)$ .

15. Find three arithmetic means between 19 and 35; and five arithmetic means between  $2\frac{2}{3}$  and  $\frac{2}{3}$ .

Ans. 23, 27, 31, and  $2\frac{1}{3}, 2, 1\frac{2}{3}, 1\frac{1}{3}, 1$ .

16. The sum of 15 terms of an arithmetic series is 600, and the common difference is 5; find the first term.      Ans. 5.

17. Given  $s = 40, a = 7, d = 2$ ; find  $n$ .

Ans.  $n = 4$ , or  $-10$ .

18. The first term of an arithmetical series is 3, and the sum of 10 terms is 165; find the progression.

Ans. 3, 6, 9, 12, &c.

19. Insert four arithmetic means between  $2\frac{1}{2}$  and  $6\frac{1}{2}$ .

Ans.  $3\frac{1}{2}, 4, 4\frac{3}{2}, 5\frac{1}{2}$ .

20. The sum of 9 terms of an arithmetic progression is 0, and the last term is  $-\frac{1}{3}$ ; find the series.

Ans.  $\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \&c.$

21. Insert three arithmetic means between  $-1$  and  $15$ .

Ans. 3, 7, and 11.

22. The sum of an arithmetic series is  $6\frac{7}{8}$ , the first term  $1\frac{1}{2}$ , and the common difference  $-\frac{1}{8}$ ; find the number of terms.

Ans. 10.

23. The third and sixth terms of an arithmetical progression are 7 and 16 respectively; find the series.

Ans. 1, 4, 7, 10, &c.

24. There are  $x$  arithmetic means between 1 and 31, and the 7th is  $\frac{5}{9}$  of the  $(x-1)$ th mean; find the number of means.

Ans. 14.

25. There are three numbers in arithmetic progression, whose sum is 10, and the product of the second and third is  $33\frac{1}{3}$ ; find them.

Ans.  $-3\frac{1}{3}, 3\frac{1}{3}, 10$ .

26. The sum of three numbers in arithmetical progression is 15, and the sum of the squares 98; find them. Ans. 2, 5, 8.

27. Find the  $n$ th term, and the sum of  $n$  terms of  $\frac{x^2 - 1}{x} + x + \frac{x^2 + 1}{x}$ , &c.

Ans.  $x + (n-2) \cdot \frac{1}{x}$ , and  $nx + \frac{1}{2}n \cdot (n-3) \cdot \frac{1}{x}$ .

28. The sum of three numbers in arithmetic progression is 24, and their product 480; find them. Ans. 6, 8, 10.

29. The first term is  $n^2 - n + 1$ , the common difference 2; find the sum of  $n$  terms. Ans.  $n^3$ .

30. Find four numbers in arithmetical progression, such that the product of the extremes shall be 27, and the product of the means 35. Ans. 3, 5, 7, 9.

31. If the  $n$ th and  $m$ th terms of an arithmetical progression be  $m$  and  $n$  respectively, find the number of terms whose sum is  $\frac{1}{2} \cdot (m + n) \cdot (m + n - 1)$ , and the last term of the series. Ans.  $m + n$ , or  $m + n - 1$ ; and 0, or 1.

32. If a steam-engine is observed to pass over 4 feet in the first second, and 88 feet in the sixtieth second of its motion, how far will it travel in the first minute, supposing its motion to be increased each second by a constant quantity?

Ans. 2760 feet.

33. Referring to the last example, find the uniform rate of increase, and the time occupied in travelling the first mile.

Ans.  $1\frac{2}{3}$  feet; and  $83\frac{2}{3}$  seconds, nearly.

34. The number of terms of an arithmetic progression is equal to  $\frac{1}{2}$  the common difference, the last term is equal to 4 times the first, and the sum of the series is equal to  $\frac{1}{2}$  the square of the first term; find the series.

Ans. 20, 32, 44, 56, 68, 80.

35. A starts from a certain place, and travels  $a$  miles the first day,  $2a$  the second,  $3a$  the third, &c.; after 4 days, B starts to overtake him, travelling  $9a$  miles per day; after how many days will he come up with him? Ans. 4.

36. A number of persons bought a field for 345*l.*, the youngest paying a certain sum, the next 5*l.* more, and so on, in arithmetic progression. The younger half took a portion of the field proportional to the sum they had subscribed, and this they agreed to divide equally, by equalizing their contribution to 22*l.* each; how many persons were there in all?

Ans. 10.

37. Find the  $n$ th term of an arithmetical progression when the sum of  $n + 1$  terms is  $(n + 1)(n + 1\frac{1}{3})$ .

Ans.  $2(n - \frac{1}{3})$ .

38. The sum of  $n$  terms of an arithmetic progression is  $pn + qn^2$ ; find the  $m$ th term.

Ans.  $p + (2m - 1)q$ .

39. Divide  $\frac{n}{7}(n + 4)$  into  $n$  parts, such that each part shall exceed the one immediately preceding by a fixed quantity.

Ans.  $\frac{4}{7}, 1, 1\frac{2}{7}, 1\frac{4}{7}, \&c.$

40. Determine the relation between  $a$ ,  $b$ , and  $c$ , that they may be the  $p$ th,  $q$ th, and  $r$ th terms of an arithmetic progression.

Ans.  $(q - r)a + (r - p)b + (p - q)c = 0$ .

41. Whereabouts in a coal shaft will the curves meet, if the radius of the roll be  $3\frac{1}{2}$  feet, the thickness of the rope  $\frac{1}{8}$  foot, and the depth of the pit 1020 feet?

Ans. 70.1273 fathoms.

## GEOMETRICAL PROGRESSION.

XXIII. When the terms of a series of quantities increase or decrease in a *common ratio*, the series is a *geometric progression*; thus 1, 3, 9, 27, 81 is a geometric progression;

whose common ratio is 3; also  $\frac{1}{3}, \frac{1}{6}, \frac{1}{12}, \frac{1}{24}, \frac{1}{48}$ , is a

geometrical progression whose common ratio is  $\frac{1}{2}$ . The

common ratio may always be found by dividing any term by the one immediately preceding it.

*Prop.* To find the  $n$ th term and the sum of any geometric progression.

Let  $a$  be the first term,  $l$  the  $n$ th,  $r$  the common ratio, and  $S$  the sum of the series; then the series will be

$$a, ar, ar^2, ar^3 \dots ar^{n-2}, ar^{n-1},$$

and it is evident that the  $n$ th term  $l = ar^{n-1}$ .

$$\text{Now } S = a + ar + ar^2 \dots + ar^{n-2} + ar^{n-1},$$

$$\therefore Sr = ar + ar^2 \dots + ar^{n-2} + ar^{n-1} + ar^n,$$

$$\therefore Sr - S = ar^n - a, \text{ by subtraction,}$$

$$\text{or } S(r - 1) = a(r^n - 1),$$

$$\therefore S = a \cdot \frac{r^n - 1}{r - 1}.$$

$$\text{Cor. 1. } \therefore S = \frac{ar^n - a}{r - 1}, \text{ and } rl = ar^n, \therefore S = \frac{rl - a}{r - 1}.$$

*Cor. 2.*  $\therefore S = \frac{ar^n}{r - 1} - \frac{a}{r - 1}$ ; if  $r$  be a proper fraction, as  $n$  increases,  $r^n$  will decrease, and when  $n$  is increased without limit,  $r^n$  will be less than any assignable magnitude,  $\therefore \frac{ar^n}{r - 1}$  may be rejected, and  $-\frac{a}{r - 1}$  or  $\frac{a}{1 - r}$  will express the *limit* of the series.

Hence, when  $r$  is a proper fraction and the series is continued ad infinitum,  $S = \frac{a}{1 - r}$ .

A *geometric mean* between two quantities = the square root of their product: thus, if  $a, m, b$  are in geometric progression,  $\frac{m}{a} = \frac{b}{m}$ ,  $\therefore m^2 = ab$ ,  $\therefore m = \sqrt{ab}$ .

### Examples.

1. Find the sum of 12 terms of the series 1, 2, 4, 8, &c.

Here  $a = 1$ ,  $r = 2$ ,  $n = 12$ ,

$$\therefore S = a \cdot \frac{r^n - 1}{r - 1} = 1 \cdot \frac{2^{12} - 1}{2 - 1} = 4096 - 1 = 4095.$$

2. Sum 6561, 2187, 729, &c., to 6 terms.

Here  $a = 6561$ ,  $r = \frac{1}{3}$ ,  $n = 6$ ,

$$\begin{aligned}\therefore S &= a \cdot \frac{r^n - 1}{r - 1} = a \cdot \frac{\left(\frac{1}{3}\right)^6 - 1}{\frac{1}{3} - 1} = a \cdot \frac{\frac{1}{729} - 1}{\frac{1}{3} - 1} \\ &= a \cdot \frac{1 - 729}{243 - 729} = a \cdot \frac{-728}{-486} = 6561 \times \frac{364}{243} \\ &= 27 \times 364 = 9828.\end{aligned}$$

3. Sum  $\frac{2}{3} - \frac{1}{3} + \frac{1}{6} - \&c.$ , to infinity.

Here  $a = \frac{2}{3}$ ,  $r = -\frac{1}{2}$

$$\therefore S = \frac{a}{1 - r} = \frac{\frac{2}{3}}{1 + \frac{1}{2}} = \frac{4}{6 + 3} = \frac{4}{9}.$$

4. Insert 4 geometric means between  $\frac{1}{8}$  and 81.

Here there are 6 terms, namely, 2 extremes and 4 means;  
 $a = \frac{1}{8}$ ,  $l = 81$ ,  $n = 6$ .

Now  $l = ar^{n-1}$ ,  $\therefore 81 = \frac{1}{8}r^5$ ,  $r^5 = 243 = 3^5$ ,  $\therefore r = 3$ .

Hence  $\frac{1}{3} \times 3 = 1$ ,  $1 \times 3 = 3$ ,  $3 \times 3 = 9$ ,  $9 \times 3 = 27$ ,  
and the means are 1, 3, 9, 27.

5. If an arithmetic mean between  $a$  and  $b$  be twice as great as a geometric mean,  $\frac{a}{b} = \frac{2 + \sqrt{3}}{2 - \sqrt{3}}$ .

$\frac{a+b}{2}$  = the arith. mean,  $\sqrt{ab}$  = the geom. mean.

$$\text{Now } \frac{a+b}{2} = 2\sqrt{ab}, \therefore a+b = 4\sqrt{a}\cdot\sqrt{b},$$

$$\frac{\sqrt{a}}{\sqrt{b}} + \frac{\sqrt{b}}{\sqrt{a}} = 4, \therefore \frac{a}{b} + 2 + \frac{b}{a} = 16,$$

$$\frac{a}{b} - 2 + \frac{b}{a} = 12,$$

$$\left. \begin{aligned} \frac{\sqrt{a}}{\sqrt{b}} - \frac{\sqrt{b}}{\sqrt{a}} &= 2\sqrt{3}, \\ \frac{\sqrt{a}}{\sqrt{b}} + \frac{\sqrt{b}}{\sqrt{a}} &= 4, \end{aligned} \right\} \therefore \begin{aligned} 2 \cdot \frac{\sqrt{a}}{\sqrt{b}} &= 4 + 2\sqrt{3}, \\ 2 \cdot \frac{\sqrt{b}}{\sqrt{a}} &= 4 - 2\sqrt{3}. \end{aligned}$$

$$\frac{\sqrt{a}}{\sqrt{b}} = 2 + \sqrt{3}, \quad \frac{\sqrt{b}}{\sqrt{a}} = 2 - \sqrt{3}.$$

$$\text{Hence } \frac{\sqrt{a}}{\sqrt{b}} \div \frac{\sqrt{b}}{\sqrt{a}} = \frac{a}{b} = \frac{2 + \sqrt{3}}{2 - \sqrt{3}}.$$

6. The sum of a series to infinity is 2, and the sum of the squares of the terms of the same series is  $\frac{4}{3}$ ; find  $a$  and  $r$ .

$$a + ar + ar^2 + ar^3 + \&c. = \frac{a}{1-r} = 2, \quad (1).$$

$$a^2 + a^2r^2 + a^2r^4 + a^2r^6 + \&c. = \frac{a^2}{1-r^2} = \frac{4}{3}, \quad (2).$$

$$\text{Dividing (2) by (1), } \frac{a}{1+r} = \frac{2}{3}, \therefore a = \frac{2}{3}(1+r).$$

Also (1)  $a = 2(1-r)$ ; and, equating these values of  $a$ ,

$$\frac{2}{3}(1+r) = 2(1-r), \quad 1+r = 3-3r,$$

$$4r = 2, \therefore r = \frac{1}{2}, \text{ and } a = 2 - 2r = 2 - 1 = 1.$$

7. If  $s = 1 + R + R^2 + R^3 + \&c.$ , to infinity, and  
 $s = 1 + r + r^2 + r^3 + \&c.$ , to infinity, prove that  
 the sum of  $1 + Rr + R^2r^2 + \&c.$ , to infinity  $= \frac{Ss}{s + s - 1}$ .

$$\text{Now, } 1 + Rr + R^2r^2 + \&c. = \frac{1}{1 - Rr}$$

$$S = \frac{1}{1 - R}, \quad s = \frac{1}{1 - r}, \quad \therefore S - SR = 1, \quad s - sr = 1,$$

$$SR = S - 1, \quad sr = s - 1, \quad \therefore R = 1 - \frac{1}{S}, \quad r = 1 - \frac{1}{s}.$$

$$\therefore Rr = 1 - \frac{1}{S} - \frac{1}{s} + \frac{1}{Ss}, \quad \therefore 1 - Rr = \frac{1}{S} + \frac{1}{s} - \frac{1}{Ss}.$$

$$\text{Hence } 1 + Rr + R^2r^2 + \&c. = \frac{1}{\frac{1}{S} + \frac{1}{s} - \frac{1}{Ss}} = \frac{Ss}{s + s - 1}.$$

$$8. \text{ Show that } 4.5212121 \&c. = 4 \frac{86}{165}.$$

$$4.5212121 \&c. = 4 + \frac{5}{10} + \frac{21}{1000} + \frac{21}{100000} + \frac{21}{10000000} + \&c.$$

$$\text{But } \frac{21}{1000} + \frac{21}{100000} + \frac{21}{10000000} + \&c.$$

$$= \frac{\frac{21}{1000}}{1 - \frac{1}{100}} = \frac{21}{1000 - 10} = \frac{21}{990} = \frac{7}{330},$$

$$\therefore 4.5212121 \&c. = \frac{45}{10} + \frac{7}{330} = \frac{1492}{330} = 4 \frac{86}{165}.$$

9. Find the sum of  $a - (a + d)x + (a + 2d)x^2 - (a + 3d)x^3 + \&c.$ , to infinity.

$$S = a - ax - dx + ax^2 + 2dx^2 - ax^3 - 3dx^3 + \&c.,$$

$$Sx = ax - ax^2 - dx^2 + ax^3 + 2dx^3 - ax^4 - 3dx^4 + \&c.,$$

$$\therefore S + Sx = a - dx + dx^2 - dx^3 + dx^4 - dx^5 + \&c.,$$

$$\text{or } (1+x)S = a - d(x - x^2 + x^3 - \&c.) = a - d \cdot \frac{x}{1+x},$$

$$\therefore S = \frac{a}{1+x} - \frac{dx}{(1+x)^2} = \frac{a + (a-d)x}{(1+x)^2}.$$

10. If  $P$  be the product,  $S$  the sum, and  $S_1$  the sum of the reciprocals of  $n$  quantities in geometrical progression; prove that  $P^2 = \left(\frac{S}{S_1}\right)^n$ .

$$\text{1st series, } S = a + ar + ar^2 + \dots ar^{n-1}, = a \cdot \frac{r^n - 1}{r - 1}.$$

2nd series,

$$\begin{aligned} S_1 &= \frac{1}{a} + \frac{1}{ar} + \frac{1}{ar^2} + \dots \frac{1}{ar^{n-1}} = \frac{1}{a} \cdot \frac{\left(\frac{1}{r}\right)^n - 1}{\frac{1}{r} - 1} \\ &= \frac{1}{a} \cdot \frac{1 - r^n}{r^{n-1} - r^n} = \frac{1}{ar^{n-1}} \cdot \frac{1 - r^n}{1 - r} = \frac{1}{ar^{n-1}} \cdot \frac{r^n - 1}{r - 1}, \\ \therefore \frac{S}{S_1} &= a^2 r^{n-1}, \quad \left(\frac{S}{S_1}\right)^n = a^{2n} r^{n(n-1)}. \end{aligned}$$

$$\text{Now } P = a \cdot ar \cdot ar^2 \dots ar^{n-1} = a^n \cdot r^{1+2+3+\dots+(n-1)}.$$

Sum the arithmetic series  $1 + 2 + 3 + \dots (n-1)$ ,

$$\begin{aligned} S' &= \frac{n'}{2} \{2a + (n' - 1)d\} = \frac{n-1}{2} \{2 + (n-2)\} = \\ &= \frac{2n - 2 + n^2 - 3n + 2}{2}, \\ &= \frac{n^2 - n}{2} = \frac{n(n-1)}{2}, \end{aligned}$$



$$\therefore P = a^n \cdot r^{\frac{n(n-1)}{2}}, \quad P^2 = a^{2n} \cdot r^{n(n-1)}.$$

$$\text{Hence, } P^2 = \left(\frac{S}{S_1}\right)^n.$$

11. The difference between two numbers = 12; and the arithmetic : to the geometric mean :: 5 : 4; find the numbers.

Let  $x$  and  $y$  be the numbers, then  $x - y = 12$ ,

$$\frac{x + y}{2} = \text{arithmetic mean, } \sqrt{xy} = \text{geometric mean,}$$

$$\frac{x + y}{2} : \sqrt{xy} :: 5 : 4, \quad x + y : 2\sqrt{xy} :: 5 : 4,$$

$$x + 2\sqrt{xy} + y : x - 2\sqrt{xy} + y :: 9 : 1, \text{ Art. XX.}$$

$$\sqrt{x} + \sqrt{y} : \sqrt{x} - \sqrt{y} :: 3 : 1,$$

$$3\sqrt{x} - 3\sqrt{y} = \sqrt{x} + \sqrt{y}, \quad 2\sqrt{x} = 4\sqrt{y},$$

$$\sqrt{x} = 2\sqrt{y}, \quad x = 4y, \quad 4y - y = 12, \quad 3y = 12.$$

$$\text{Ans. } y = 4, \quad x = 16.$$

$$12. \text{ If } S = 1 + \frac{3}{2} + \frac{5}{4} + \frac{7}{8} + \&c., \text{ to infinity,}$$

$$\text{and } S_1 = 1 - \frac{3}{2} + \frac{5}{4} - \frac{7}{8} + \&c., \text{ to infinity,}$$

$$S : S_1 :: 27 : 1.$$

$$\left. \begin{aligned} S &= 1 + \frac{3}{2} + \frac{5}{4} + \frac{7}{8} + \frac{9}{16} + \&c. \\ \frac{S}{2} &= \frac{1}{2} + \frac{3}{4} + \frac{5}{8} + \frac{7}{16} + \&c. \end{aligned} \right\} \text{ Subtract.}$$

$$\therefore \frac{S}{2} = 1 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 2 + \frac{1}{1 - \frac{1}{2}} = 2 + 1 = 3,$$

$$\therefore S = 6.$$

$$\left. \begin{aligned} S_1 &= 1 - \frac{3}{2} + \frac{5}{4} - \frac{7}{8} + \frac{9}{16} - \&c. \\ \frac{S_1}{2} &= \frac{1}{2} - \frac{3}{4} + \frac{5}{8} - \frac{7}{16} + \&c. \end{aligned} \right\} \text{Add.}$$

$$\therefore \frac{3S_1}{2} = 1 - 1 + \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32} - \&c.$$

$$= \left( \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \&c. \right) - \left( \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \&c. \right)$$

$$= \frac{\frac{1}{2}}{1 - \frac{1}{4}} - \frac{\frac{1}{4}}{1 - \frac{1}{4}} = \frac{2}{3} - \frac{1}{3} = \frac{1}{3};$$

$$\therefore S_1 = \frac{2}{3}, \quad \therefore S : S_1 :: 6 : \frac{2}{3}, \quad \therefore 3 : \frac{1}{9}, \quad \therefore 27 : 1.$$

13. From a vessel containing 10 gallons of brandy, 1 gallon was drawn out, and a gallon of water poured into the vessel; a gallon of the mixture was then drawn out, and another gallon of water poured in. Now the like process being repeated 10 times, it is required to find how much brandy remained in the vessel, supposing the two fluids were thoroughly mixed each time?

There are 10 gallons of brandy at first;

$\frac{1}{10}$  is drawn;

there are then 9 gallons of brandy,

and 1 gallon of water poured in;

$\frac{9}{10}$  gallon of brandy is drawn,

$$9 - \frac{9}{10} = \frac{81}{10} \text{ gallons of brandy left in,}$$

$$\frac{19}{10} \text{ gallons water left in;}$$

$$\frac{81}{100} \text{ gallons brandy drawn.}$$

Hence  $1, \frac{9}{10}, \frac{81}{100}, \&c.,$  to 10 terms = quantity of brandy drawn.

$$\text{Here } a = 1; r = \frac{9}{10}, n = 10,$$

$$\therefore s = \frac{a(r^n - 1)}{r - 1} = \frac{\left(\frac{9}{10}\right)^{10} - 1}{\frac{9}{10} - 1},$$

$$= \frac{.9^{10} - 1}{.9 - 1} = \frac{1 - .3486784401}{1 - .9},$$

$$= \frac{.6513215599}{.1} = 6.513215599 \text{ gallons drawn.}$$

Subtracting 6.513215599 from 10, we have

$$3.486784401 \text{ gallons. Ans.}$$

14. The sum of three numbers in geometrical progression is 7, and the sum of their reciprocals is  $\frac{7}{4}$ ; find them.

Let  $\frac{x}{y}$  be the first term,  $y$  the common ratio,

$$\text{then } \frac{x}{y} + x + xy = 7, \quad \text{and } \frac{y}{x} + \frac{1}{x} + \frac{1}{xy} = \frac{7}{4},$$

$$\left. \begin{array}{l} \frac{1}{y} + 1 + y = \frac{7}{x} \\ y + 1 + \frac{1}{y} = \frac{7}{4}x \end{array} \right\} \quad \therefore \frac{7}{4}x = \frac{7}{x},$$

$$\left. \begin{array}{l} \frac{1}{y} + 1 + y = \frac{7}{x} \\ y + 1 + \frac{1}{y} = \frac{7}{4}x \end{array} \right\} \quad x^2 = 4, \quad \therefore x = 2.$$

$$\therefore y + 1 + \frac{1}{y} = \frac{7}{2}, \quad y - \frac{5}{2} = -\frac{1}{y},$$

$$y^2 - \frac{5}{2}y = -1, \quad y^2 - \frac{5}{2}y + \frac{5}{4} = \frac{25}{16} - 1 = \frac{9}{16},$$

$$y - \frac{5}{4} = \pm \frac{3}{4}, \quad \therefore y = \frac{5 \pm 3}{4} = 2;$$

$\therefore 1, 2, 4$ , are the numbers.

15. The sum of four numbers in geometrical progression is 40, and the sum of their squares 820; find them.

Let  $x, xy, xy^2, xy^3$  be the numbers,

then  $x + xy + xy^2 + xy^3 = 40$ , or  $x(1 + y + y^2 + y^3) = 40$ ,  
 $x^2 + x^2y^2 + x^2y^4 + x^2y^6 = 820$ , or  $x^2(1 + y^2 + y^4 + y^6) = 820$ ,  
 or  $x\{1 + y^2 + y(1 + y^2)\} = 40$ ,  $x(1 + y)(1 + y^2) = 40$ , (1),  
 $x^2\{1 + y^2 + y^4(1 + y^2)\} = 820$ ,  $x^2(1 + y^2)(1 + y^4) = 820$ , (2).

Squaring (1) and dividing by (2), we have

$$\frac{x^2(1 + y)^2(1 + y^2)^2}{x^2(1 + y^2)(1 + y^4)} = \frac{1600}{820}, \quad \text{or} \quad \frac{(1 + y)^2 \cdot (1 + y^2)}{1 + y^4} = \frac{80}{41}.$$

This equation reduced gives

$$y^2 + \frac{1}{y^2} - \frac{82}{39} \left( y + \frac{1}{y} \right) = \frac{82}{39},$$

whence  $y = 3$ ,  $\therefore x = 1$ ,  $\therefore 1, 3, 9, 27$  are the numbers.

### Examples.

1. The sum of  $3 + 6 + 12 + \&c.$  to 6 terms  $= 189$ .
2. The sum of  $5 + 20 + 80 + \&c.$  to 5 terms  $= 1705$ .
3. The sum of  $\frac{8}{2} + 1 + \frac{2}{3} + \&c.$  to 6 terms  $= 4\frac{17}{162}$ .
4. The 8th term of the series  $9, -6, 4, \&c.$   $= -\frac{128}{243}$ .
5. The 6th term of the series  $3, \frac{1}{2}, \frac{1}{12}, \&c.$   $= \frac{1}{2592}$ .

6. The sum of  $1 - 2 + 2^2 - \&c.$  to 10 terms  $= -341$ .
7. Determine the 5th term and the sum of 5 terms of  $21 - 3 + \frac{3}{7} - \&c.$  Ans.  $\frac{3}{343}$ , and  $18 \frac{129}{843}$ .
8. The 5th term of the series  $-\frac{1}{2}, \frac{1}{8}, -\frac{2}{9}, \&c. = -\frac{8}{81}$ .
9. Find expressions for the  $n$ th term and the sum of  $n$  terms of  $\frac{4}{5} + 2 + 5 + \&c.$  Ans.  $\frac{5^{n-2}}{2^{n-3}}$ , and  $\frac{1}{15} \left( \frac{5^n - 2^n}{2^{n-3}} \right)$ .
10. Sum  $\frac{3}{5} + \frac{1}{2} + \frac{5}{12} + \&c.$  to 8 terms. Ans.  $\frac{1288991}{466560}$ .
11. Find the limit of the series  $\frac{1}{4} - \frac{1}{16} + \frac{1}{64} - \&c.$   $\frac{1}{5}$ .
12. Find the limit of the series  $1 - \frac{1}{2} + \frac{1}{4} - \&c.$   $\frac{2}{3}$ .
13. Find the limit of the series  $2 - 1\frac{1}{3} + \frac{8}{9} - \&c.$   $1\frac{1}{5}$ .
14. Find the limit of the series  $-3\frac{1}{5} + 1\frac{3}{5} - \frac{4}{5} + \&c.$   
Ans.  $-2\frac{2}{15}$ .
15. Insert three geometric means between  $\frac{1}{2}$  and 128.  
Ans.  $\pm 2, 8, \pm 32$ .
16. Find two geom. means between 5 and 1080, and three geom. means between 9 and  $\frac{1}{9}$ . Ans. 30, 180; and 3, 1,  $\frac{1}{3}$ .
17. The sum of an infinite geometric series is 2, and the sum of its first two terms is  $1\frac{1}{2}$ ; find the series.  
Ans.  $3 - \frac{3}{2} + \frac{3}{4} - \&c.$
18. The product of three numbers in geometrical progression is 64, and the sum of the cubes is 584; find them.  
Ans. 2, 4, 8.

19. If the sum of a geometrical series continued ad infinitum be twice as great as the sum of  $n$  terms, prove that the common ratio is  $\left(\frac{1}{2}\right)^{\frac{1}{n}}$ .

20. The sum of the arithmetic and geometric means of two numbers is  $18\frac{1}{2}$ , and if the geometric be subtracted from the arithmetic mean, the remainder will  $1\frac{1}{2}$ ; find the numbers.

Ans. 8 and 12.

21. A leveret is 100 yards before a greyhound; they start in the same direction, and the greyhound runs 100 times as fast as the leveret; when will the greyhound overtake the leveret?

Ans. When the hound has run  $101\frac{1}{99}$  yards.

22. The population of a country increases annually in geometric progression, and in four years was raised from 10,000 to 14,641 souls; by what part of itself was it annually increased?

Ans.  $\frac{1}{10}$ .

23. In a geometric progression, if the  $(p+q)$ th term  $= m$  and the  $(p-q)$ th term  $= n$ , show that the  $p$ th term  $=$

$\sqrt{mn}$ , and the  $q$ th term  $= m \left(\frac{n}{m}\right)^{\frac{p}{2q}}$ . Also, if  $P$  be the  $p$ th term and  $Q$  the  $q$ th term, show that the  $n$ th term  $= \left(\frac{P^{n-q}}{Q^{n-p}}\right)^{\frac{1}{p-q}}$ .

24. Determine  $m$  and  $n$  in terms of  $a$  and  $b$ , so that  $\frac{ma+nb}{m+n}$  may be the arithmetic mean between  $m$  and  $n$ , and the geometric mean between  $a$  and  $b$ .

25. Given  $S$  the sum, and  $s^2$  the sum of the squares, of the terms of an infinite geometrical progression; show that its sum to  $n$  terms  $= S \left\{ 1 - \left( \frac{S^2 - s^2}{S^2 + s^2} \right)^n \right\}$ .

## HARMONICAL PROGRESSION.

XXIV. When the first of three quantities is to the third as the difference of the first and second is to the difference of the second and third, the three quantities are in *harmonic proportion*.

$$(1) \frac{2xy}{x+y} = 11 - (x+y),$$

$$\therefore \frac{4x^2y^2}{(x+y)^2} = 121 - 22(x+y) + x^2 + 2xy + y^2 \quad \left. \vphantom{\frac{4x^2y^2}{(x+y)^2}} \right\} \text{Subtract.}$$

$$(2) \frac{4x^2y^2}{(x+y)^2} = 49 \quad \begin{array}{r} -x^2 \\ -y^2 \end{array}$$

$$0 = 72 - 22(x+y) + 2x^2 + 2xy + 2y^2,$$

$$\therefore x^2 + xy + y^2 = 11(x+y) - 36 \quad \left. \vphantom{x^2 + xy + y^2} \right\} \text{Subtract.}$$

$$(1) x^2 + 4xy + y^2 = 11(x+y)$$

$$3xy = 36, \therefore xy = 12,$$

$$\begin{array}{r} x^2 + xy + y^2 - 11(x+y) = -36 \\ xy = 12 \end{array}$$

$$x^2 + 2xy + y^2 - 11(x+y) = -24, \therefore x+y=8.$$

Whence  $x=6$ ,  $\therefore y=2$ ,  $\therefore 2, 8, 6$  are the numbers.

### Examples.

1. Insert two harmonic means between 2 and 4. Ans.  $2\frac{2}{3}, 3$ .

2. Insert three harmonic means between 4 and  $1\frac{1}{2}$ .

Ans. 3,  $2\frac{2}{3}$ , and 2.

3. Find the arithmetic, geometric, and harmonic means between  $3\frac{1}{2}$  and  $1\frac{1}{2}$ .

Ans.  $2\frac{7}{10}, 2\frac{1}{4}, 2\frac{1}{10}$ .

4. Insert two harmonic means between  $\frac{1}{2}$  and  $\frac{1}{11}$ .  $\frac{1}{4}$  and  $\frac{1}{9}$ .

5. Continue to three terms each way the series  $1\frac{1}{2}, 2\frac{1}{7}, 3\frac{1}{2}$ .

Ans.  $\frac{1}{9}, \frac{1}{8}, 1\frac{2}{3}, \dots, 15, -7\frac{1}{2}, -8$ .

6. Find two numbers whose difference is 8, and the harmonic mean between them  $1\frac{1}{3}$ . Ans. 9 and 1, or  $\frac{4}{3}$  and  $-7\frac{1}{3}$ .

7. An harmonic mean between two numbers is 2, and the sum of the extremes =  $4\frac{4}{5}$ ; find the series. Ans.  $2\frac{2}{5}, 2, 1\frac{1}{5}$ .

8. The sum of the arithmetic and harmonic means of two numbers is  $12\frac{5}{7}$ , and if the harmonic be subtracted from the arithmetic mean, the remainder will be  $1\frac{2}{7}$ ; find the numbers.

Ans. 4 and 10.

9. If the geometric mean between  $x$  and  $y$  : the harmonic mean  $:: m : n$ , then  $x : y :: m + \sqrt{(m^2 - n^2)} : m - \sqrt{(m^2 - n^2)}$ .

10. Find the sum of  $n$  terms of  $1 + 1 + 2 + 3 + 5 + 8 + \&c.$ , any term being the sum of the two preceding.

$$\text{Ans. } \frac{1}{\sqrt{5}} \left\{ \left( \frac{1 + \sqrt{5}}{2} \right)^{n+2} - \left( \frac{1 - \sqrt{5}}{2} \right)^{n+2} \right\} - 1.$$

## CHAPTER IX.

### BINOMIAL SURDS.

#### *Propositions.*

XXV. 1. The square root of a quantity cannot be partly rational, and partly a quadratic surd.

If possible, let  $\sqrt{x} = a + \sqrt{y}$ , then  $x = a^2 + 2a\sqrt{y} + y$ ,  
 $\therefore x - y - a^2 = 2a\sqrt{y}$ ,  $\therefore \frac{x - y - a^2}{2a} = \sqrt{y}$ ; or, a rational quantity = an irrational quantity, which is impossible.

2. If  $a + \sqrt{x} = b + \sqrt{y}$  be an equation between rational quantities and quadratic surds, then  $a = b$ , and  $\sqrt{x} = \sqrt{y}$ .

For  $\sqrt{x} = b - a + \sqrt{y}$ , and if  $b - a$  be not  $= 0$ ,  $\sqrt{x}$  would be partly rational and partly a quadratic surd, which is impossible, by Proposition 1,  $\therefore b - a = 0$ , or  $b = a$ ,  
 $\therefore \sqrt{x} = \sqrt{y}$ .

3. The product of two dissimilar surds is irrational. If possible, let  $\sqrt{x} \times \sqrt{y} = ax$ , then  $xy = a^2 x^2$ ,  $y = a^2 x$ ,  
 $\therefore \sqrt{y} = a\sqrt{x}$ ; that is,  $\sqrt{y}$  and  $\sqrt{x}$  may have the same surd factor, which is contrary to the supposition.

4. To extract the square root of a binomial consisting of a rational quantity and a quadratic surd.



Let the given binomial be  $a + \sqrt{b}$ .

Assume  $\sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y}$ ; square both sides,

$$a + \sqrt{b} = x + y + 2\sqrt{xy},$$

$$\therefore x + y = a, \quad 2\sqrt{xy} = \sqrt{b}, \quad (\text{Prop. 2.})$$

$$x^2 + 2xy + y^2 = a^2, \quad 4xy = b,$$

$$x^2 - 2xy + y^2 = a^2 - b, \quad \therefore x - y = \sqrt{a^2 - b},$$

$$\frac{x + y}{2x} = \frac{a}{a + \sqrt{a^2 - b}},$$

$$2y = a - \sqrt{a^2 - b},$$

$$\therefore x = \frac{a + \sqrt{a^2 - b}}{2}, \quad y = \frac{a - \sqrt{a^2 - b}}{2},$$

$$\therefore \sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y} =$$

$$\pm \sqrt{\frac{a + \sqrt{a^2 - b}}{2}} \pm \sqrt{\frac{a - \sqrt{a^2 - b}}{2}}.$$

### Examples.

1. Extract the square root of  $7 \pm 4\sqrt{3}$ .

Assume  $\sqrt{7 \pm 4\sqrt{3}} = \sqrt{x} \pm \sqrt{y}$ ,

$$\text{then } 7 \pm 4\sqrt{3} = x + y \pm 2\sqrt{xy},$$

$$\therefore x + y = 7, \quad 2\sqrt{xy} = 4\sqrt{3},$$

$$x^2 + 2xy + y^2 = 49, \quad 4xy = 48,$$

$$\therefore \left. \begin{aligned} x^2 - 2xy + y^2 &= 1, & \therefore x - y &= 1 \\ x + y &= 7 \end{aligned} \right\} \therefore \begin{aligned} x &= 4, \\ y &= 3, \end{aligned}$$

$$\therefore \sqrt{7 \pm 4\sqrt{3}} = \sqrt{x} \pm \sqrt{y} = 2 \pm \sqrt{3}.$$

2. Extract the fourth root of  $49 + 20\sqrt{6}$ .

Assume  $\sqrt{49 + 20\sqrt{6}} = \sqrt{x} + \sqrt{y}$ ,

$$\text{then } 49 + 20\sqrt{6} = x + y + 2\sqrt{xy},$$

$$\therefore x + y = 49, \quad 2\sqrt{xy} = 20\sqrt{6},$$

$$x^2 + 2xy + y^2 = 2401, \quad 4xy = 2400,$$

$$\therefore x^2 - 2xy + y^2 = 1, \quad \therefore x - y = 1 \quad \left. \begin{array}{l} \therefore x = 25 = 5^2, \\ x + y = 49 \end{array} \right\} \therefore y = 24 = 4 \times 6,$$

$$\therefore \sqrt{49 + 20\sqrt{6}} = \sqrt{x} + \sqrt{y} = 5 \pm 2\sqrt{6}$$

Again, assume  $\sqrt{5 \pm 2\sqrt{6}} = \sqrt{x} \pm \sqrt{y}$ ,

$$\text{then } 5 \pm 2\sqrt{6} = x + y \pm 2\sqrt{xy},$$

$$\therefore x + y = 5, \quad 2\sqrt{xy} = 2\sqrt{6},$$

$$x^2 + 2xy + y^2 = 25, \quad 4xy = 24,$$

$$x^2 - 2xy + y^2 = 1, \quad x - y = 1 \quad \left. \begin{array}{l} \therefore x = 3, \\ x + y = 5 \end{array} \right\} \therefore y = 2,$$

$$\therefore \sqrt[4]{49 + 20\sqrt{6}} = \sqrt{5 \pm 2\sqrt{6}} = \sqrt{x} \pm \sqrt{y} = \sqrt{3} \pm \sqrt{2}.$$

3. Find the eighth root of  $-1$ .

The square root of  $-1$  is  $\sqrt{-1}$  or  $0 + \sqrt{-1}$ .

Assume that  $\sqrt{x} + \sqrt{y} = \sqrt{0 + \sqrt{-1}}$ .

$$x + y + 2\sqrt{xy} = 0 + \sqrt{-1},$$

$$x + y = 0, \quad 2\sqrt{xy} = \sqrt{-1},$$

$$x^2 + 2xy + y^2 = 0$$

$$4xy = -1$$

---


$$x^2 - 2xy + y^2 = 1$$

$$x - y = 1$$

$$x + y = 0$$

---


$$2x = 1, \quad \therefore x = \frac{1}{2}, \quad \sqrt{x} = \sqrt{\frac{1}{2}},$$

$$2y = -1, \quad \therefore y = -\frac{1}{2}, \quad \sqrt{y} = \sqrt{-\frac{1}{2}},$$

$$\therefore \sqrt{x} + \sqrt{y} = \sqrt{\frac{1}{2}} + \sqrt{-\frac{1}{2}},$$

$$\therefore \sqrt{\sqrt{-1}} = \sqrt{\frac{1}{2}} + \sqrt{-\frac{1}{2}}.$$

Assume  $\sqrt{x} + \sqrt{y} = \sqrt{\sqrt{\frac{1}{2}} + \sqrt{-\frac{1}{2}}},$

$$x + y + 2\sqrt{xy} = \sqrt{\frac{1}{2}} + \sqrt{-\frac{1}{2}},$$

$$x + y = \sqrt{\frac{1}{2}}, \quad 2\sqrt{xy} = \sqrt{-\frac{1}{2}},$$

$$x^2 + 2xy + y^2 = \frac{1}{2}$$

$$4xy = -\frac{1}{2}$$

---


$$x^2 - 2xy + y^2 = 1$$

$$x - y = 1$$

$$x + y = \sqrt{\frac{1}{2}}$$

---


$$2x = 1 + \sqrt{\frac{1}{2}}, \therefore x = \frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}},$$

$$2y = \sqrt{\frac{1}{2}} - 1, \therefore y = \frac{1}{2}\sqrt{\frac{1}{2}} - \frac{1}{2},$$

$$\therefore \sqrt{x} = \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}}, \quad \sqrt{y} = \sqrt{\frac{1}{2}\sqrt{\frac{1}{2}} - \frac{1}{2}},$$

$$\therefore \sqrt{x} + \sqrt{y} = \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}} + \sqrt{\frac{1}{2}\sqrt{\frac{1}{2}} - \frac{1}{2}},$$

$$\therefore \sqrt[8]{-1} = \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}} + \sqrt{\frac{1}{2}\sqrt{\frac{1}{2}} - \frac{1}{2}} \\ = .9239 + .3827 \sqrt{-1} \text{ nearly.}$$

*Examples.*

1. Extract the square root of  $2 + \sqrt{3}$ ; of  $8 + 2\sqrt{7}$ ; and of  $4 - \sqrt{7}$ .

Ans.  $\sqrt{\frac{1}{2}} + \sqrt{\frac{3}{2}}$ ;  $1 + \sqrt{7}$ ; and  $\sqrt{\frac{7}{2}} - \sqrt{\frac{1}{2}}$ .

2. Of  $8 - 2\sqrt{15}$ ;  $1 - 4\sqrt{-3}$ ;  $1 + \sqrt{1 - m^2}$ ; and  $3\sqrt{3} + 2\sqrt{6}$ .

Ans.  $\sqrt{5} - \sqrt{3}$ ;  $2 - \sqrt{-3}$ ;  $\sqrt{\frac{1+m}{2}} + \sqrt{\frac{1-m}{2}}$ ; and  $\sqrt[4]{3}(1 + \sqrt{2})$ .

3. Extract the fourth root of  $14 + 8\sqrt{3}$ ; and of  $\frac{17}{3} - 4\sqrt{2}$ .

$$\text{Ans. } \frac{\sqrt{3} + 1}{\sqrt[4]{2}}; \text{ and } \sqrt[4]{\frac{1}{3}}(1 - \sqrt{2}).$$

## CHAPTER X.

### INDETERMINATE COEFFICIENTS.

XXVI. Let  $A + Bx + Cx^2 + \dots = a + bx + cx^2 + \dots$  be an equation which holds true for any value whatever of  $x$ ; then the coefficients of like powers of  $x$  shall be equal to each other; that is,  $A = a$ ,  $B = b$ ,  $C = c$ , &c.

For  $A - a + Bx - bx + Cx^2 - cx^2 + \&c. \dots = 0$ ,

$$\text{or } A \sim a + (B \sim b)x + (C \sim c)x^2 + \dots = 0.$$

Now, if  $A \sim a$  be not equal to 0, let it be equal to some fixed quantity  $M$ , then

$$(B \sim b)x + (C \sim c)x^2 + \dots = M;$$

and  $\therefore A$  and  $a$  are invariable quantities,  $\therefore A \sim a$  or  $M$  is *invariable*. But  $\therefore M$  may have various values dependent upon the variations of  $x$ ,  $\therefore M$  is *variable*; that is,  $M$  is both variable and invariable, which is impossible;

$$\therefore A \sim a = 0, \text{ or } A = a.$$

Again,  $B \sim b + (C \sim c)x + \dots = 0$ ,

$$\therefore \text{similarly } B = b, \text{ and } C = c, \&c.$$

$$\text{If } A + Bx + Cx^2 + \dots + A'y + B'xy + \dots + A''y^2 + \dots = a + bx + cx^2 + \dots + a'y + b'xy + \dots + a''y^2 + \dots$$

and if some fixed value be given to  $x$  while  $y$  is variable, it may be shown, as above, that

$$A = a, B = b, C = c, A' = a', B' = b', C' = c', \&c.$$

### Examples.

1. Resolve  $\frac{2x}{(x^2 + 1)(x^2 + 3)}$  into its partial fractions.

$$\text{Assume } \frac{2x}{(x^2+1)(x^2+3)} = \frac{Ax}{x^2+1} + \frac{Bx}{x^2+3} =$$

$$\frac{Ax^3 + 3Ax + Bx^3 + Bx}{(x^2+1)(x^2+3)},$$

$$\therefore 2x = (A+B)x^3 + (3A+B)x,$$

$$\therefore 2 = 3A + B, \text{ and } (A+B)x^3 = 0, \quad \therefore A = -B,$$

$$\therefore 2 = -3B + B = -2B, \quad \therefore B = -1,$$

$$3A = 2 - B = 2 + 1 = 3, \quad \therefore A = 1.$$

$$\text{Hence } \frac{2x}{(x^2+1)(x^2+3)} = \frac{x}{x^2+1} - \frac{x}{x^2+3}.$$

2. Let  $y^3 - 3y + x = 0$ ; find the value of  $y$  in a series of ascending powers of  $x$ .

$$\left. \begin{array}{l} \text{Assume } y = Ax + Bx^3 + Cx^5 + Dx^7 + \&c. \\ \text{then } y^3 = \quad \quad \quad A^3x^3 + 3A^2Bx^5 + 3A^2Cx^7 + \&c. \\ \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad + 3AB^2x^7 + \&c. \\ - 3y = -3Ax - 3Bx^3 - 3Cx^5 - 3Dx^7 - \&c. \\ + x = +1x. \end{array} \right\} = 0.$$

And, equating coefficients of like powers of  $x$ , we have

$$-3A = -1, \text{ or } A = \frac{1}{3}; \quad A^3 = 3B, \text{ or } B = \frac{A^3}{3} = \frac{1}{3^4};$$

$$3A^2B = 3C, \text{ or } C = A^2B = \frac{1}{3^6}, \&c.,$$

$$\therefore y = \frac{x}{3} + \frac{x^3}{3^4} + \frac{x^5}{3^6} + \&c.$$

If we had assumed  $y = Ax + Bx^2 + Cx^3 + Dx^4 + \&c.$ , we should have found that the coefficients of the even powers of  $x$  would be equal to 0.

*Examples.*

1. Find the fractions whose sum is  $\frac{x^3}{(x^2-1)(x-2)}$ .

$$\text{Ans. } \frac{4}{3(x-2)} - \frac{1}{2(x-1)} + \frac{1}{6(x+1)}.$$

2. Expand  $\frac{1+2x}{1-x-x^2}$  in a series of ascending powers of  $x$ .

$$\text{Ans. } 1 + 3x + 4x^2 + 7x^3 + 11x^4 + \&c.$$

3. Prove that  $\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \&c.$

4. Prove that  $1^2 + 2^2 + 3^2 + 4^2 + \&c. + n^2 = \frac{n(n+1)(2n+1)}{6}.$

## CHAPTER XI.

## LOGARITHMS.

XXVII. If  $a^x = n$ ,  $x$  is called the *logarithm* of the number  $n$  to the base  $a$ . In the common system of logarithms,  $a = 10$ , and  $0, 1, 2, 3, 4 \dots x$ , are the logarithms of the numbers  $1, 10, 100, 1000 \dots 10^x$ ; that is, the numbers being in *geometric* progression, their logarithms are in *arithmetic* progression.

Logarithms are calculated by finding  $x$  from the equation  $10^x = n$ , in series involving the powers of  $x$ , and are published in books of Logarithmic Tables.

*Propositions.*

1. The sum of the logarithms of two or more numbers is the logarithm of their product.

Let  $x = \log n$ , and  $y = \log n'$ ; then,  $a$  being the base,

$$a^x = n, a^y = n'; \therefore a^x a^y = n n', \text{ or } a^{x+y} = n n'.$$

And similarly for more numbers.

2. The difference of the logarithms of two numbers is the logarithm of their quotient.

$$\text{For } a^x = n, a^y = n'; \therefore \frac{a^x}{a^y} = \frac{n}{n'}, \text{ or } a^{x-y} = \frac{n}{n'}.$$

3. The logarithm of  $n^m = m \log n$ .

$$\text{For } \because n = a^x, \therefore n^m = a^{mx};$$

$$\therefore \log n^m = mx = m \log n.$$

4. The logarithm of  $\sqrt[m]{n} = \frac{\log n}{m}$ .

$$\text{For } \because n = a^x, \therefore n^{\frac{1}{m}} = a^{\frac{x}{m}};$$

$$\therefore \log n^{\frac{1}{m}} = \frac{x}{m} = \frac{\log n}{m}.$$

Hence the *multiplication* and *division* of numbers may be performed by the *addition* and *subtraction* of logarithms, and the *involution* or *evolution* of numbers, by *multiplying* or *dividing* their logarithms by the indices of the powers or roots required.

#### Examples.

1. If  $x = ab$ , then  $\log x = \log a + \log b$ .
2. If  $y = 6ac$ , then  $\log y = \log 6 + \log a + \log c$ .
3. If  $x = \frac{m}{r}$ , then  $\log x = \log m - \log r$ .
4. If  $y = a^3$ , then  $\log y = 3 \log a$ .
5. If  $z = \sqrt[4]{c}$ , then  $\log z = \frac{1}{4} \log c$ .

6. If  $s = a \cdot \frac{r^n - 1}{r - 1}$ , find a logarithmic expression for the value of  $n$ .

$$s(r - 1) = ar^n - a, \quad ar^n = s(r - 1) + a,$$

$$r^n = \frac{s(r - 1) + a}{a}, \quad \therefore n \cdot \log r = \log \{s(r - 1) + a\} - \log a,$$

$$\therefore n = \frac{\log \{s(r - 1) + a\} - \log a}{\log r}.$$

7. If  $5^x = 400$ , find  $x$ .

$$x \log 5 = \log 400,$$

$$\therefore x = \frac{\log 400}{\log 5} = \frac{2.602060}{.698970} = 3.72 \text{ nearly.}$$

The logarithms of 400 and 5 are taken from the tables of Brigg's Logarithms.

The number standing before the decimal part of a logarithm is called the *characteristic*, and it is always less by unity than the number of digits in the integral part of the number: thus the number 400 consisting of three digits, the characteristic of its logarithm is 2. Hence, in tables of logarithms, it is usual to omit the characteristic.

The logarithm of 1752 is 3.243534.

... of 175.2 is 2.243534.

... of 17.52 is 1.243534.

... of 1.752 is 0.243534.

... of .1752 is — 1.243534.

... of .01752 is — 2.243534.

&c.

&c.



## CHAPTER XII.

## BINOMIAL THEOREM.

XXVIII. The Binomial Theorem enables us to raise a binomial to any power without the process of repeated multiplication.

*Prop.* Let it be required to find the  $n$ th power of the binomial  $a + x$ .

$$a + x = a \left( 1 + \frac{x}{a} \right), \therefore (a + x)^n = a^n \left( 1 + \frac{x}{a} \right)^n.$$

Assume  $\frac{x}{a} = y$ , and that

$(1 + y)^n = A + By + Cy^2 + Dy^3 + \dots Py^p + \&c. \dots (1)$ ,  
the coefficients A, B, C, D, &c. being independent of  $y$ .

Squaring both sides of this equation, we have

$$(1 + y)^{2n} = \left. \begin{aligned} &A^2 + 2ABy + 2ACy^2 + 2ADy^3 + 2AEy^4 + \&c. \\ &+ B^2y^2 + 2BCy^3 + 2BDy^4 + \&c. \\ &+ C^2y^4 + \&c. \end{aligned} \right\} \dots (2).$$

Now  $\therefore (1 + y)^{2n} = (1 + 2y + y^2)^n = \{1 + (2y + y^2)\}^n$ ,  
 $\therefore$  by assumption (1),

$$\begin{aligned} (1 + y)^{2n} &= A + B(2y + y^2) + C(2y + y^2)^2 + D(2y + y^2)^3 + \&c. \\ &= A + 2By + By^2 + 4Cy^3 + Cy^4 + \&c. \\ &\quad + 4Cy^2 + 8Dy^3 + 16Ey^4 + \&c. \end{aligned} \dots (3)$$

And  $\therefore$  the series (2) and (3) are each  $= (1 +$

$$\therefore A^2 + 2ABy + (2AC + B^2)y^2 + (2AD + 2BC)y^3 -$$

$$A + 2By + (B + 4C)y^2 + (4C + 8D)y^3 + \dots \&c. \log c.$$

Hence, by Art. XXVI.,  $A^2 = A$ ,  $\therefore A = 1$ ;

$$2AB = 2B, \therefore B = B;$$

$$2AC + B^2 = B + 4C, \therefore C = \frac{B^2 - B}{2} = \frac{B(B - 1)}{1 \cdot 2}.$$

$$2AD + 2BC = 4C + 8D,$$

$$\therefore D = \frac{2C(B - 2)}{6} = \frac{B(B - 1)(B - 2)}{1 \cdot 2 \cdot 3}; \&c.$$

$$1 + By + \frac{B(B-1)}{1.2}y^2 + \frac{B(B-1)(B-2)}{1.2.3}y^3 + \&c. \dots (4).$$

Putting  $My^2$  for all the terms after  $By$ , we have

$$(1+y)^n = 1 + By + My^2, \quad n \log(1+y) = \log(1 + By + My^2), \quad \text{Art. XXVII.}$$

and, if we assume  $\log(1+y) = a + by + cy^2 + \&c.$ ,  $a, b, c, \&c.$  being independent of  $y$ ; then

$$\log\{1 + (By + My^2)\} = a + b(By + My^2) + c(By + My^2)^2 + \&c.$$

$$\therefore na + nby + ncy^2 + \&c. = a + bBy + bMy^2 + \&c.,$$

$$\therefore \text{by Art. XXVI., } na = a, \therefore a = 0, \quad nb = bB, \therefore B = n.$$

Hence, substituting  $n$  for  $B$  in (4), we have

$$(1+y)^n = 1 + ny + \frac{n(n-1)}{1.2}y^2 + \frac{n(n-1)(n-2)}{1.2.3}y^3 + \&c.,$$

$$\text{or, } \left(1 + \frac{x}{a}\right)^n = 1 + n\frac{x}{a} + \frac{n(n-1)x^2}{1.2a^2} + \frac{n(n-1)(n-2)x^3}{1.2.3a^3} + \&c.,$$

$$\therefore (a+x)^n = a^n \left(1 + \frac{x}{a}\right)^n = a^n + na^{n-1}x + \frac{n(n-1)}{1.2}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}a^{n-3}x^3 + \&c.,$$

which is the Binomial Theorem of Sir Isaac Newton.

*Cor. 1.* If  $x$  be negative, its even powers will be positive, and its odd powers negative; hence

$$(a-x)^n = a^n - na^{n-1}x + \frac{n(n-1)}{1.2}a^{n-2}x^2 - \frac{n(n-1)(n-2)}{1.2.3}a^{n-3}x^3 + \&c.$$

$$\text{Cor. 2. If } a=1, \quad (1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 + \&c.,$$

$$\therefore (a+b+c)^n = \{(a+b)+c\}^n,$$

$$\text{and } (a+b+c+d)^n = \{(a+b)+(c+d)\}^n,$$

the Binomial Theorem may be applied to the expansion of powers of a polynomial.

The above proof is general, and consequently applies to the cases of  $n$  being an integer, a fraction, positive or negative.

On inspecting this theorem, it appears that

The powers of  $a$  decrease and those of  $x$  increase by unity in each succeeding term.

The *first* term is the  $n$ th power of the first term of the given binomial.

The *second* term is found by multiplying the index of the first by the first with its index diminished by unity, and also by the second term of the binomial.

The coefficient of any succeeding term is found by multiplying the coefficient and index of  $a$  in the preceding term together, and dividing by the number of terms already set down.

In applying this theorem, it will appear that

When  $n$  is an integer, the number of terms in the expansion is  $n+1$ ; and the coefficients of any two terms equidistant from the extremes are equal to each other.

When  $n$  is a fraction the series does not terminate.

### Examples.

1. Expand  $(a+x)^6$ .

$$\begin{aligned} (a+x)^6 &= a^6 + 6a^5x + \frac{6.5}{2}a^4x^2 + \frac{6.5.4}{2.3}a^3x^3 \\ &+ \frac{6.5.4.3}{2.3.4}a^2x^4 + \frac{6.5.4.3.2}{2.3.4.5}a^1x^5 + \frac{6.5.4.3.2.1}{2.3.4.5.6}x^6 \\ &= a^6 + 6a^5x + 15a^4x^2 + 20a^3x^3 + 15a^2x^4 + 6ax^5 + x^6. \end{aligned}$$

Practically, the coefficients after the second term are found thus,

$$\frac{6.5}{2} = 15, \quad \frac{15.4}{3} = 20,$$

and after the middle term they may be written in an inverse order.

$$2. (a-x)^5 = a^5 - 5a^4x + 10a^3x^2 - 10a^2x^3 + 5ax^4 - x^5.$$

$$3. (1-x)^7 = 1 - 7x + 21x^2 - 35x^3 + 35x^4 - 21x^5 + 7x^6 - x^7.$$

$$4. (2x+1)^5 = 32x^5 + 80x^4 + 80x^3 + 40x^2 + 10x + 1.$$

$$5. \frac{1}{(c+x)^2} = \frac{1}{c^2} \left( 1 - \frac{2x}{c} + \frac{3x^2}{c^2} + \frac{4x^3}{c^3} + \&c. \right).$$

$$6. \sqrt[3]{1-x^3} = 1 - \frac{x^3}{3} - \frac{x^6}{9} - \frac{5x^9}{81} - \&c.$$

## EXPONENTIAL THEOREM.

$$\text{XXIX. } \because a = 1 + \overline{a-1}, \quad \therefore a^x = (1 + \overline{a-1})^x = \{(1 + \overline{a-1})^n\}^{\frac{x}{n}};$$

Hence, by the Binomial Theorem, (Cor. 2.),

$$\begin{aligned} a^x &= \left\{ 1 + n(a-1) + \frac{n(n-1)}{2}(a-1)^2 + \right. \\ &\quad \left. \frac{n(n-1)(n-2)}{2 \cdot 3}(a-1)^3 + \&c. \right\}^{\frac{x}{n}} = \\ &= \left\{ 1 + [(a-1) - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \&c.] n + \right. \\ &\quad \left. Bn^2 + Cn^3 + \&c. \right\}^{\frac{x}{n}}, \end{aligned}$$

where B, C, &c. contain powers of  $(a-1)$  only.

Let now  $(a-1) - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \&c. = A$ , then

$$\begin{aligned} a^x &= \{1 + An + Bn^2 + Cn^3 + \&c.\}^{\frac{x}{n}} = \\ &= 1 + \frac{x}{n}(An + Bn^2 + \&c.) + \\ &\quad \frac{\frac{x}{n}\left(\frac{x}{n} - 1\right)}{2}(An + Bn^2 + \&c.)^2 + \&c. = \end{aligned}$$

$$1 + x(A + Bn + \&c.) + \frac{x(x-n)}{2}(A + Bn + \&c.)^2 + \&c.$$

Now since  $n$  may have any value whatever, let  $n = 0$ , then

$$a^x = 1 + Ax + \frac{A^2 x^2}{1.2} + \frac{A^3 x^3}{1.2.3} + \&c.$$

If  $e$  be that value of  $a$  which makes  $A = 1$ ,

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \&c., \text{ the Exponential Theorem.}$$

If  $x = 1$ ,

$e = 1 + \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} + \&c. = 2.71828, \&c.$ , which is the base of the system of logarithms used by Napier, the inventor of logarithms.

$$\text{XXX. } \therefore a + b = \frac{a}{1 - \frac{b}{a+b}},$$

$$\therefore (a+b)^n = \frac{a^n}{\left(1 - \frac{b}{a+b}\right)^n} = a^n \left(1 - \frac{b}{a+b}\right)^{-n}.$$

$$\therefore (a+b)^n = a^n \left\{ 1 + n \cdot \frac{b}{a+b} + \frac{n(n+1)}{2} \cdot \left(\frac{b}{a+b}\right)^2 + \&c. \right\} (\text{Art. XXVIII.})$$

This is Colson's Theorem.

## CHAPTER XIII.

### VARIATIONS, PERMUTATIONS, AND COMBINATIONS.

XXXI. The different arrangements which can be made of any number of quantities, taking a certain number at a time, are called their *Variations*.

Thus, if  $a, b, c$  be taken two together, their variations will be  $ab, ba, ac, ca, bc, cb$ .

If all the quantities are taken together, their variations are called *Permutations*.

Thus the permutations of  $a, b, c$  are  $abc, acb, bac, bca, cab, cba$ .

XXXII. The number of variations of  $n$  different things, taken  $r$  together, is  $n(n-1)(n-2) \dots \{n-(r-1)\}$ .

Let  $a, b, c, d, \&c.$ , be the  $n$  things; then the number of variations which can be made, taking them singly, is  $n$ .

Let  $n-1$  of these things, namely  $b, c, d, \&c.$  be taken singly, then the number of their variations is  $n-1$ : and if  $a$  be placed before each, we shall have  $n-1$  variations of  $n$  things, taken two together, in which  $a$  stands first; similarly we shall have  $n-1$  such variations, in which  $b$  stands first; and similarly for all the  $n$  things, hence there will be on the whole  $n(n-1)$  variations of  $n$  things taken *two* together.

Again, taking  $n-1$  of these things, namely  $b, c, d, \&c.$ , their variations, taken *two* together, will be  $n(n-1)(n-2)$ ; and proceeding as before, there will be on the whole  $(n-1)(n-2)$  variations of  $n$  things taken *three* together.

Similarly their variations, taken *four* together, will be  $n(n-1)(n-2)(n-3)$ . Hence, if  $V_1, V_2, V_3, \&c. V_r$  denote the variations of  $n$  things, taken 1, 2, 3, &c.  $r$  together, we have

$$V_1 = n, \quad V_2 = n(n-1), \quad V_3 = n(n-1)(n-2), \&c.$$

$$V_r = n(n-1)(n-2)(n-3) \dots \{n-(r-1)\}.$$

*Cor.*  $\therefore$  the permutations ( $p$ ) of  $n$  things are their variations, taken *all together*; by writing  $n$  for  $r$  we shall have

$$\begin{aligned} p &= n(n-1)(n-2) \dots \{n-(n-2)\} \{n-(n-1)\} \\ &= n(n-1)(n-2) \dots 2.1 = 1.2.3 \dots n. \end{aligned}$$

XXXIII. When  $a$  and  $b$  are different, their permutations are  $ab, ba$ , but when  $a = b$ , they become  $aa$ .

Let  $a$  recur  $p$  times,  $b$   $q$  times,  $c$   $r$  times, &c., and  $P$  be the number of permutations required. Then if all the  $a$ 's be changed into different letters, they will form  $1.2.3 \dots p$

permutations, and out of *each* of the  $P$  permutations we should form  $1.2.3 \dots p$  permutations, so that there would be  $P$  times  $1.2.3 \dots p$  permutations. Similarly if all the  $b$ 's be changed to different letters, they would form  $1.2.3 \dots q$  permutations, and therefore there would be  $P.(1.2.3 \dots p.1.2.3 \dots q)$  permutations. Now, when all the quantities have become different, the number of permutations is  $1.2.3 \dots n$ . (Art. XXXII., *Cor.*)

$$\therefore P.(1.2.3 \dots p.1.2.3 \dots q.1.2.3 \dots r. \&c.) = 1.2.3 \dots n,$$

$$\therefore P = \frac{1.2.3 \dots n}{1.2.3 \dots p.1.2.3 \dots q.1.2.3 \dots r. \&c.}$$

XXXIV. Suppose there are  $n$  things,  $a, b, c, d, \&c.$ , and that they may be repeated; then, if they be taken two together, we shall have  $n$  variations where  $a$  stands first,  $n$  where  $b$  stands first,  $n$  where  $c$  stands first,  $\&c.$  And since there are  $n$  things, the entire numbers of variations will be  $n \times n$ , or  $n^2$ .

If they be taken three together, since each of them may stand before each variation, the entire number of variations will be  $n \times n^2$ , or  $n^3$ .

In the same manner, if they be taken  $n$  together, the entire number of variations will be  $n^n$ .

And if it be required to determine the total number of variations with repetitions of  $n$  things taken one at a time, two at a time,  $\&c.$ , and all together, we must take the sum of the geometric series  $n + n^2 + n^3 + \dots n^n$ .

XXXV. The different collections that can be made of a number of things, taking a certain number together without regarding their order, are called their *Combinations*.

Thus the combinations of  $a, b, c$ , taken two together, are  $ab, ac, bc$ .

Each combination will supply as many corresponding variations as the number of things it contains admits of permutations.

Each combination of  $r$  things supplies  $1.2.3 \dots r$  variations of  $r$  things; hence, if  $C_r$  be the number of combinations of  $n$  things, taken  $r$  together

$$C_r \cdot (1 \cdot 2 \cdot 3 \dots r) = V_r = n(n-1)(n-2) \dots \{n-(r-1)\},$$

$$\therefore C_r = \frac{n(n-1)(n-2) \dots \{n-(r-1)\}}{1 \cdot 2 \cdot 3 \dots r}.$$

*Cor.* If 1, 2, 3, 4, &c. be successively put for  $r$ , we shall have

$$C_1 = \frac{n}{1}, C_2 = \frac{n(n-1)}{1 \cdot 2}, C_3 = \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}, \&c.$$

$$\therefore C_1 + C_2 + C_3 + \&c. = n + \frac{n(n-1)}{1 \cdot 2} + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} + \&c.$$

$$\text{But } (1+x)^n = 1 + nx + \frac{n(n-1)}{1 \cdot 2} x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} x^3 + \&c.$$

(Art. XXVIII.)

$$\therefore (1+1)^n = 2^n = 1 + n + \frac{n(n-1)}{1 \cdot 2} + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} + \&c.$$

$$\therefore 2^n - 1 = n + \frac{n(n-1)}{1 \cdot 2} + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} + \&c.$$

$\therefore C_1 + C_2 + C_3 + \&c. = 2^n - 1 =$  the sum of all the combinations that can be made of  $n$  things taken 1, 2, 3, &c.  $n$  together.

XXXVI. The number of combinations of  $n$  sets of things, containing respectively  $p, q, r$ , &c. things, one being taken out of each set for each combination, is equal to the continued product of  $p, q, r$ , &c.

First, if there be two sets, containing  $p$  and  $q$  things respectively, the number of combinations will be  $pq$ .

Then, if there be another set, containing  $r$  things, each of them being combined with  $pq$  combinations, the number of combinations will be  $pqr$ , &c.

*Cor.* If  $p=q=r=\&c.$ , the remainder of combinations  $= p^n$ .

XXXVII. If of  $n$  things  $r$  be taken,  $n-r$  things will be left, and for every different set of  $r$  things taken, there will be a different set containing  $n-r$  remaining; therefore, the number of the former sets will constantly be the same as that of the latter.

Hence, the number of combinations of  $n$  things taken  $r$  together is the same as the number of combinations of  $n$



things taken  $n - r$  together. This may be expressed thus,  ${}^nC_r = {}^nC_{n-r}$ .

XXXVIII. To find the number of combinations of two sets of things containing respectively  $m$  and  $n$  things, by taking  $r$  out of one set and  $s$  out of the other for each combination.

$${}^mC_r \times {}^nC_s = \frac{m(m-1)\dots(m-r+1)}{1.2\dots r} \times \frac{n(n-1)(n-s+1)}{1.2\dots s}.$$

(Art. XXXV.)

XXXIX. To find what number  $r$  out of  $n$  things must be taken together, so that the number of combinations formed may be a maximum.

Since  $C_r$  is obtained by multiplying  $C_{r-1}$ , by  $\frac{n-r+1}{r}$ , or  $\frac{n+1}{r} - 1$ , the quantities  $C_1, C_2$ , &c. will increase continually, each term upon the preceding, so long as their factor  $> 1$ ; hence  $C_r$  will be the greatest for the greatest value of  $r$  which allows of this, or if  $n+1 > 2r$ , that is, when  $r$  is the integer next  $< \frac{1}{2}(n+1)$ .

If  $n$  be even,  $r = \frac{1}{2}n$ , if  $n$  be odd, and  $\therefore \frac{1}{2}(n+1)$  an integer,  $r = \frac{1}{2}(n+1) - 1 = \frac{1}{2}(n-1)$ ; but in this case, since  $C_r = C_{n-r}$  (Art. XXXVII.), the number taken  $\frac{1}{2}(n-1)$  together = number taken  $n - \frac{1}{2}(n-1)$  or  $\frac{1}{2}(n+1)$  together, and  $\therefore r$  may be either  $\frac{1}{2}(n-1)$  or  $\frac{1}{2}(n+1)$ .

### Examples.

1. How many changes can be rung with 7 bells out of 10?

By Variations, Art. XXXII,

$$V_r = n(n-1)(n-2)\dots\{n-(r-1)\};$$

and  $\therefore$  there are 10 bells,  $n = 10$ ,

and  $\therefore$  they are taken 7 at a time,  $r = 7$ , and  $r-1 = 6$ ,

$$\therefore n - (r-1) = 10 - 6 = 4.$$

Hence  $V_7 = 10.9.8.7.6.5.4 = 604800$  changes.

2. How often can 6 boys change their places in a class, so as not to preserve the same order?

By Permutation, Art. XXXII., Cor.,  $p = 1.2.3\dots n$ ,

$$\therefore p = 1.2.3.4.5.6 = 720 \text{ times.}$$

3. In how many ways may the word enunciation be written?

$$\text{By Permutation, Art. XXXIII., } P = \frac{1.2.3 \dots n}{1.2..p.1.2..q.1.2..r};$$

and  $\therefore$  there are 11 letters, of which 3 are  $n$ 's and 2 are  $i$ 's,  
 $\therefore n = 11, p = 3, q = 2$ .

$$\text{Hence, } P = \frac{1.2.3.4.5.6.7.8.9.10.11}{1.2.3.1.2} = 3326400 \text{ ways.}$$

4. Into how many different triangles may a decagon be divided by drawing lines from the angular points?

The number of triangles will be equal to the number of lines that can be drawn by connecting 7 at a time of the 10 angles with each angle; that is, it will be equal to the number of combinations of 10 angles taken 7 together. Combination (Art. XXXV.),

$$\begin{aligned} C &= \frac{n(n-1)(n-2) \dots \{n-(r-1)\}}{1.2.3 \dots r}, \\ &= \frac{10.9.8.7.6.5.4}{1.2.3.4.5.6.7} = 120. \end{aligned}$$

5. The total number of combinations of  $2n$  things : total number of  $n$  things :: 129 : 1; find  $n$ .

Combinations (Art. XXXV. Cor.),  $2^n - 1$ ; write  $2n$  for  $n$ , then

$$2^{2n} - 1 = \text{total number of combinations of } 2n \text{ things,}$$

$$2^n - 1 = \dots \dots \dots n \dots$$

$$\frac{2^{2n} - 1}{2^n - 1} = \frac{129}{1}, \text{ or } \frac{(2^n + 1)(2^n - 1)}{2^n - 1} = \frac{129}{1},$$

$$2^n + 1 = 129, \quad 2^n = 128, \quad 2^n = 2^7, \quad \therefore n = 7. \text{ Ans.}$$

6. How many different collections and words may be formed of 5 letters of the alphabet, 2 vowels and 3 consonants being taken together?

Combinations (Arts. XXXVIII., XXXV.),

$${}^nC_r \times {}^nC_s = \frac{n(n-1) \dots (n-r+1)}{1.2 \dots r} \times \frac{n(n-1) \dots (n-s+1)}{1.2 \dots s}$$

Here  $m = 19$  consonants,  $n = 5$  vowels  $\left. \begin{array}{l} m-r+1=17, \\ r=3 \quad ,, \quad s=2 \quad ,, \end{array} \right\} \begin{array}{l} n-s+1=4, \end{array}$

$$\therefore {}^mC_r \times {}^nC_s = \frac{19.18.17}{1.2.3} \times \frac{5.4}{1.2} = 969 \times 10 = 9690 \text{ collections.}$$

But since each collection consists of 5 letters, each may be permuted 5.4.3.2.1 different ways; for (Permutation, Art. XXXII., Cor.),

$$p = n(n-1) \cdot (n-2) \dots 3 \cdot 2 \cdot 1.$$

Here  $n = 5$ ,

$\therefore p = 5.4.3.2.1 = 120$  permutations of the 9690 collections,

$$\therefore 9690 \times 120 = 1162800 \text{ words. Ans.}$$

7. A telegraph has  $m$  arms, and each arm is capable of  $n$  distinct positions; find the total number of signals that can be made with the telegraph.

Combination (Art. XXXV., Cor.). First find the number of combinations of  $m$  arms taken 1 at a time, 2 at a time, &c.,

$${}^mC_1 = m, {}^mC_2 = \frac{m(m-1)}{1.2}, {}^mC_3 = \frac{m(m-1)(m-2)}{1.2.3}, \&c.;$$

$$\therefore {}^mC_1 + {}^mC_2 + {}^mC_3 + \&c. = m + \frac{m(m-1)}{1.2} + \frac{m(m-1)(m-2)}{1.2.3}$$

+ &c. = number of combinations of the  $m$  arms, taken at first singly, then 2 at a time, then 3 at a time, &c. But as each arm is capable of  $n$  distinct positions, the entire number of combinations with changes of position =

$$mn + \frac{m(m-1)}{1.2}n^2 + \frac{m(m-1)(m-2)}{1.2.3}n^3 + \&c. =$$

$$(1+n)^m - 1. \text{ Art. XXVIII., Cor. 2.}$$

7. How many changes may be rung on 5 bells? Ans. 120.

8. How many different numbers can be made out of one nit, two 2's, three 3's, and four 4's, supposing all the figures to be in every number? Ans. 12600.

9. How many changes can be rung with 4 bells out of 8?

Ans. 1344.

10. Given the number of variations of  $2n + 1$  things,  $n - 1$  together: number of variations of  $2n - 1$ ,  $n$  together,  $:: 3 : 5$ ; determine  $n$ .  
Ans. 4.

11. How many different hands may a person hold at the game of whist; that is, how many changes can be made of 13 cards out of 52?  
Ans. 635013559600.

12. A company of soldiers consists of 50 men, and 4 of them are selected every night to mount guard; on how many nights can a different selection of the 4 sentinels be made?  
Ans. 230300.

13. What is the total number of combinations of 16 things, taken 1, 2, 3, &c. at a time?  
Ans. 65535.

14. The number of variations of  $m + n$  things, 2 together is 56, and of  $m - n$  things is 12; find the number of combinations of  $m$  things,  $n$  together.  
Ans. 15.

15. Find the number of permutations with repetitions (that is, allowing things which recur to be combined as if they were different) of 12 things, taken 5 together.  
Ans. 248832.

16. Show that the number of different combinations of  $n$  things taken 1, 2, 3, &c.  $n$  together, of which  $p$  are of one sort,  $q$  of another,  $r$  of another, &c., is  $(p + 1)(q + 1)(r + 1) \dots 1$ .

17. There are 4 companies of soldiers, consisting of 40, 42, 45 and 50 men respectively; how many selections of 4 men can be made by taking one out of each company?  
Ans. 3780000.

## CHAPTER XIV.

### COMPOUND INTEREST AND ANNUITIES.

- XL. Let  $P$  be the *principal*, or sum lent,  
 $M$  the *amount*, or sum of principal and interest,  
 $r$  the interest of £1 for the first period of time,  
 $R = 1 + r$ , the amount of £1 for that period,  
 $n$  the number of periods.

Then, at the end of the first period  $R$  will be the principal, and  $\therefore$  £1 invested amounts to  $R$  pounds,

$$\therefore 1 : R :: R : R^2 = \text{amount of } \text{£}1 \text{ in 2 periods,}$$

$$\text{similarly } 1 : R^2 :: R : R^3 = \dots\dots\dots 3 \dots\dots$$

and so on.  $\therefore R^n =$  amount of £1 in  $n$  periods of time of equal length; and when  $P$ £ is the given principal, the amount will be  $P$  times as great as when £1 is the principal.

$$\therefore M = PR^n, \log M = \log P + n \log R \text{ (Art. XXVII.)}$$

By this last equation, any three of the quantities  $M, P, n, R$  being given, the fourth may be found.

XLI. Let  $A$  be an annuity, or sum of money annually payable,  $R$  the amount of £1 in one year; then,  $A$  is the amount of the annuity at the end of the first year,  $1 : A :: R : RA =$  its amount at the end of the second year,  $\therefore A + RA = A(1 + R)$  is the sum due at the end of the second year; similarly  $1 : A(1 + R) :: R : A(R + R^2)$  is the amount of the two payments at the end of the third year,  $\therefore A + A(R + R^2) = A(1 + R + R^2)$  is the sum due at the end of the third year; similarly  $A(1 + R + R^2 + R^3)$  is the sum due at the end of the fourth year; and generally  $A(1 + R + R^2 + \dots R^{n-1}) = A \cdot \frac{R^n - 1}{R - 1} = M$ , the amount of the annuity, at compound interest, in  $n$  years.

$$\log M = \log A + \log(R^n - 1) - \log(R - 1).$$

XLII. Let  $V$  be the *present value* of the annuity  $A$ , then

$$VR^n = A \cdot \frac{R^n - 1}{R - 1}, \quad \therefore V = A \cdot \frac{1 - R^{-n}}{R - 1},$$

$$\log V = \log A + \log(1 - R^{-n}) - \log(R - 1).$$

*Cor.* If the annuity be *perpetual*,  $n$  is infinite,

$$\therefore R^{-n} = \frac{1}{R^n} = 0,$$

and  $V = \frac{A}{R - 1} =$  the present value of a *perpetual annuity* of  $A$ £ per annum.

XLIII. Let the annuity  $A$  commence after  $m$  years, and continue for  $n$  years, then its present value will be the difference between its present value for  $m + n$  years, and its present value for  $m$  years, or

$$V = A \cdot \frac{1 - R^{-(m+n)}}{R - 1} - A \cdot \frac{1 - R^{-m}}{R - 1} = \frac{AR^{-m} - AR^{-m-n}}{R - 1}$$

$$= \frac{AR^{-m}(1 - R^{-n})}{R - 1},$$

$$\log V = \log A - m \log R + \log(1 - R^{-n}) - \log(R - 1).$$

*Cor.* If the annuity be perpetual,  $n$  is infinite,  $\therefore R^{-n} = 0$ , and  $V = \frac{AR^{-m}}{R - 1}$  = the present value of a *deferred annuity*, to continue for ever.

XLIV. Let  $N$  denote any number, whole, fractional, or mixed; let  $\lambda N$  denote its logarithm, and  $\pi N$  the complement of that logarithm, so that  $\lambda \frac{1}{N} = -\lambda N = \pi N$ ; then the formulæ adapted to logarithmic computation in the last four articles may be put into a more convenient shape by writing  $\lambda$  for  $\log$ , and  $\pi$  for  $-\log$ .

*Examples.*

1. What sum laid up now and improved at compound interest during 12 years, will amount to 5000*l.*, the rate of interest being  $4\frac{1}{2}$  per cent., the interest being payable yearly?

By Art. XL.  $\log P = -n \log R + \log M$ , or

$$\lambda P = n\pi R + \lambda M$$

$$= 12\pi(1.045) + \lambda 5000$$

$$\text{Now } \pi 1.045 = 1.9808837$$

$$\times n = 12$$

$$\lambda 5000 = 3.6989700$$

$$\lambda P = 3.4695744$$

$$\therefore P = 2948.31836 = \text{£}2948 \text{ } 6s. \text{ } 4\frac{1}{2}d. \text{ } \text{Ans.}$$

2. To what sum will an annuity of £100 amount in 20 years, at compound interest, at  $4\frac{1}{2}$  per cent., interest payable yearly?

By Art. XLI.

$$\begin{aligned}\log M &= \log (R^n - 1) + \log A - \log (R - 1), \text{ or} \\ \lambda M &= \lambda (R^n - 1) + \lambda A + x (R - 1), \\ &= \lambda (1.045^{20} - 1) + \lambda 100 + x \cdot 045.\end{aligned}$$

$$\text{Now } \lambda 1.045 = .0191163$$

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$$.3823260$$

the number to which is  $2.4117150$   
 $- 1$

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$$1.4117150$$

the log of which is  $.1497470$   
 $\lambda 100 = 2$

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$$2.1497470$$

$x \cdot 045$   $1.3467875$

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$$\lambda M = 3.4965345,$$

$$\therefore M = 3137.144 = \text{£}3137 \text{ 2s. } 10\frac{1}{2}d. \text{ Ans.}$$

If the interest were payable half-yearly,  $n$  would be 40,  
 $r$  would be  $\frac{.045}{2} = .0225$ ,  $\therefore R$  would be 1.0225.

If the interest were payable quarterly,  $n$  would be 80, and  
 $R$  1.01125.

3. What is the value of the lease of an estate for 21 years, the clear annual rental being £135, allowing 5 per cent. compound interest?

By Art. XLII.

$$\begin{aligned}\log V &= \log (1 - R^{-n}) + \log A - \log (R - 1), \text{ or} \\ \lambda V &= \lambda (1 - R^{-n}) + \lambda A + x (R - 1) \\ &= \lambda (1 - 1.05^{-21}) + \lambda 135 + x \cdot 05.\end{aligned}$$

$$\text{Now } \lambda R^{-n} = \lambda \frac{1}{R^n} = \lambda R^n = \lambda 1.05^{21} = 1.9788107$$

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$$\begin{array}{r} 9788107 \\ 19676214 \end{array}$$

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21

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$$1.5550247,$$

$$\therefore R^{-n} = .3689422, \quad 1 - R^{-n} = .6410578,$$

$$\lambda .6410578 = 1.8068973$$

$$\lambda .135 = 2.1803388$$

$$\lambda .05 = 1.3010300$$

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$$\lambda V = 3.2382611$$

$$\therefore V = 1730.8565 = £1730 \text{ } 17s. \text{ } 1\frac{1}{2}d. \text{ Ans.}$$

4. Find the compound interest of £800 for 9 years at 5 per cent. per annum, the interest being payable yearly.

Ans. £441 1s. 3d.

5. Find the amount of an annuity of £356 per annum, payable half-yearly for 9 years, allowing compound interest at 6 per cent. per annum.

Ans. £4167 15s. 4½d.

6. What is the present value of an annuity of £70 per annum, payable quarterly for five years, allowing compound interest at 5 per cent. per annum?

Ans. £307 19s. 8½d.

7. Find the present value of a deferred annuity of £1000, to commence after the expiration of 5 years, and then to continue for 20 years, allowing compound interest at 5 per cent.

Ans. £9764 9s. 4½d.

8. If a lease for 55½ years cost £100, what annual rent ought the purchaser to receive, that he may get 5½ per cent. for his money?

Ans. £5 16s.



9. What is the difference between the value of a freehold estate, or perpetual annuity of £100 per annum, and that of a leasehold-estate of £100 per annum, to continue 60 years?

Ans. The freehold is worth £107 1s. 4½d. more than the leasehold.

10. What sum ought to be paid for the reversion of an annuity of £50 for 14 years after the next 7, that the purchaser may make 5 per cent. of his money? Ans. £351 10s. nearly.

11. The sum of £518 6s. being placed out at compound interest for 3 years, amounts to £600; find the rate of interest.

Ans. 5 per cent.

12. In what time will a sum of money *double* itself, at 4 per cent. compound interest?

Ans. In 17·6 years.

13. Suppose a person to place out annually the sum of £20 for 40 successive years, what would the whole amount to at the end of that time, at 5 per cent. compound interest?

Ans. £2536 16s.

14. Find the present value of an annuity of £40, to continue 5 years, allowing compound interest at 5 per cent.

Ans. £173 3s. 7d.

15. What must be paid for an estate whose annual rental is £79 4s. that the purchaser may make 4½ per cent. of his money?

Ans. £1760.

16. A person places  $P$  pounds at interest, and adds to his capital at the end of every year  $\frac{1}{m}$ th part of the interest for that year; what is the amount at the expiration of  $n$  years?

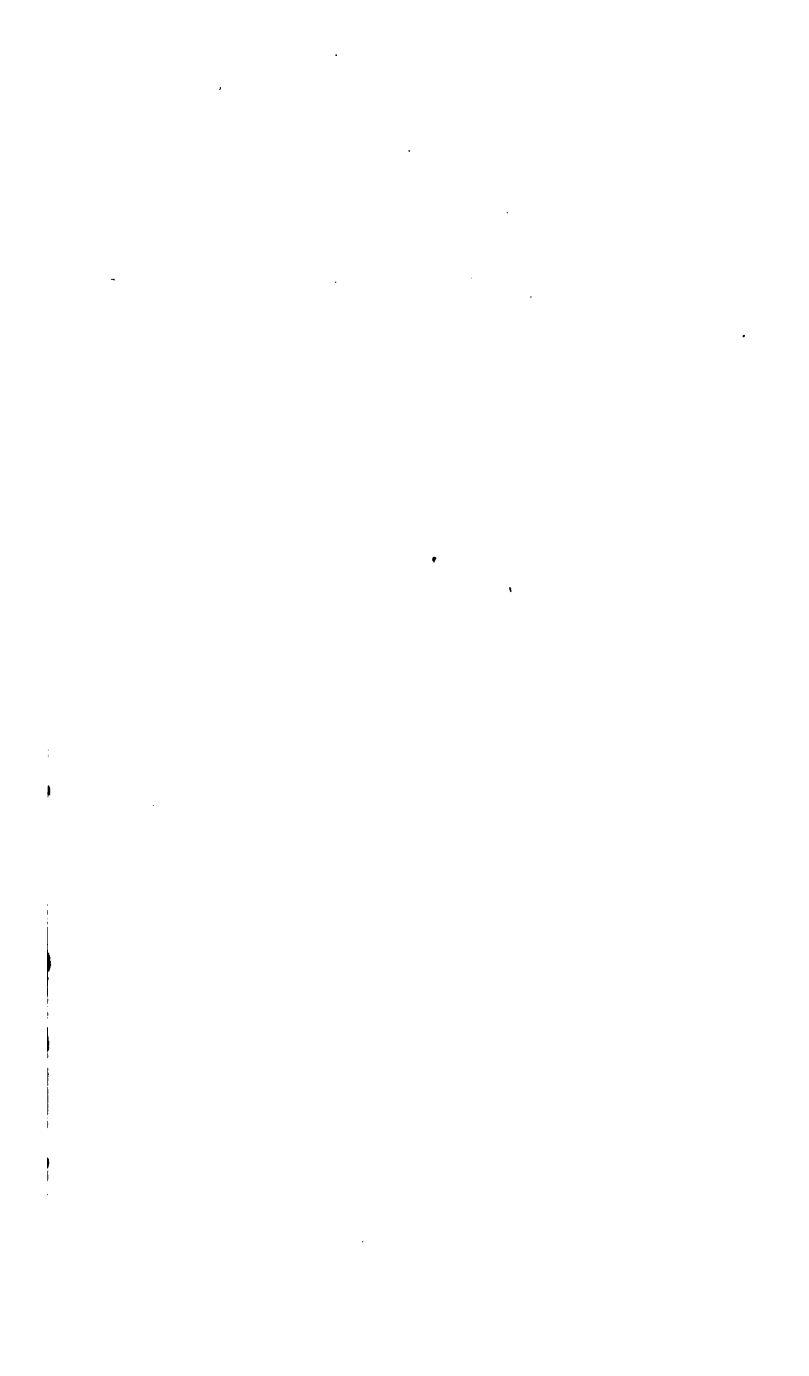
Ans.  $P \left( \frac{r}{m} + r + 1 \right)^n$ .

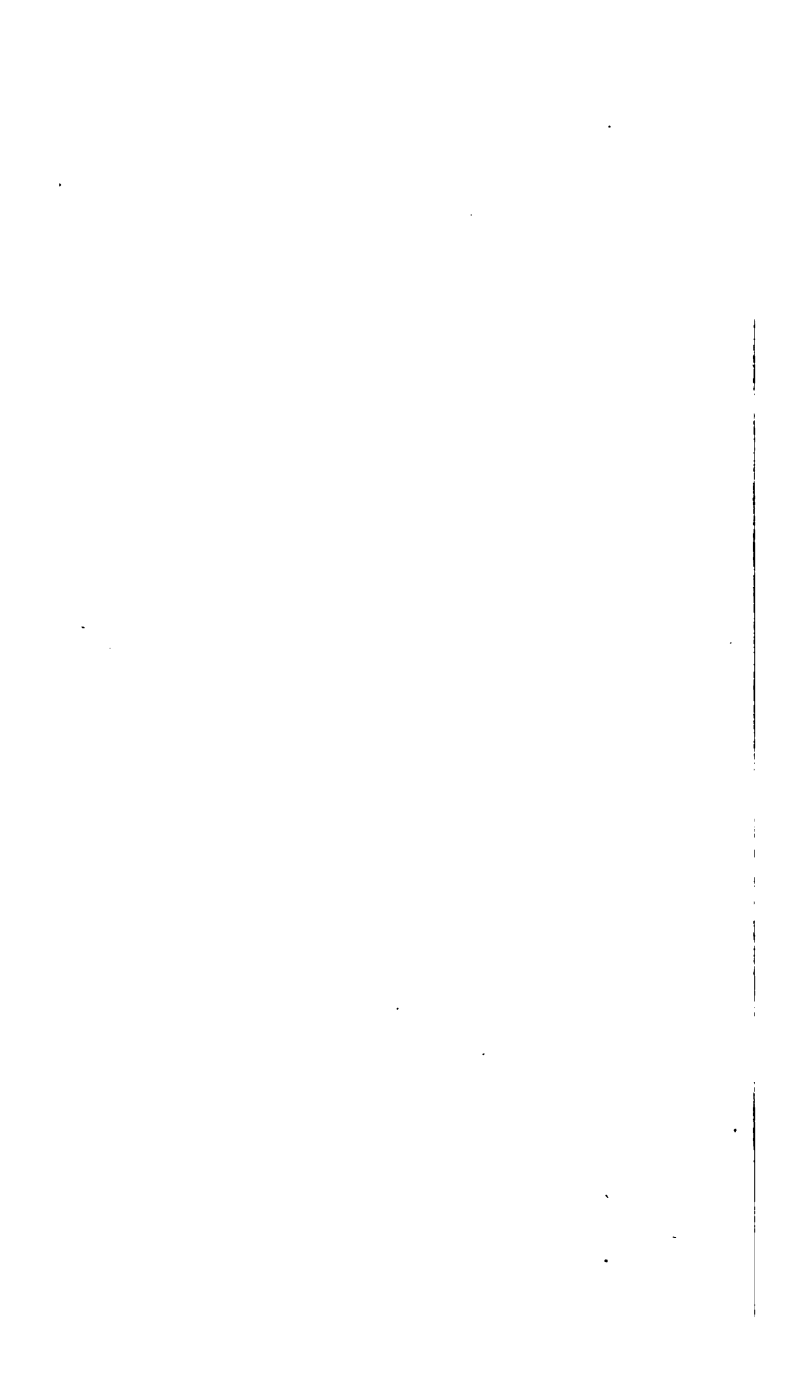
17. Find the present value of an annuity of £20, to commence 10 years hence, and then to continue 11 years, allowing 3½ per cent. compound interest.

Ans. £118 7s. 3½d.

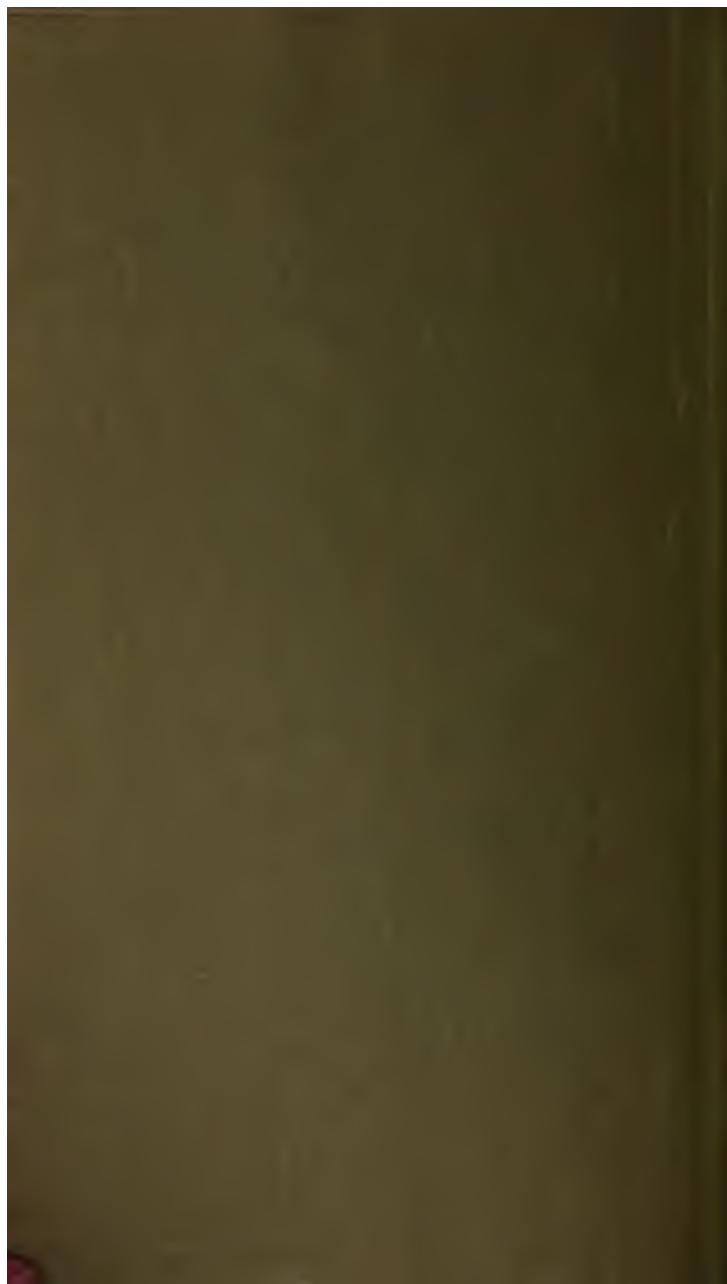
18. A person leaves to his two sons, A and B, equal shares of an estate producing £1000 per annum, but A proposes that B shall take both shares, and allow him an equivalent annuity for 20 years; what annuity ought B to allow A, reckoning interest at 4½ per cent.?

Ans. £854 2s. 4d.









MAR 30 1951

